Minimal velocity estimates and soft mode bounds for the massless spin-boson model

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Abstract

We consider generalised versions of the spin-boson model at small coupling. We assume the spin (or atom) to sit at the origin $0 \in \mathbb{R}^d$ and the propagation speed v_p of free bosons to be constant, i.e. independent of momentum. In particular, the bosons are massless. We prove detailed bounds on the mean number of bosons contained in the ball $\{|x| \leq v_p t\}$. In particular, we prove that, as $t \to \infty$, this number tends to an asymptotic value that can be naturally identified as the mean number of bosons bound to the atom in the ground state. Physically, this means that bosons that are not bound to the atom, are travelling outwards at a speed that is not lower than v_p, hence the term 'minimal velocity estimate'. Additionally, we prove bounds on the number of emitted bosons with low momentum (soft mode bounds). This paper is an extension of our earlier work in [4]. Together with the results in [4], the bounds of the present paper suffice to prove asymptotic completeness, as we describe in [2].

1 Model and result

This paper provides technical tools to prove asymptotic completeness for some models of quantum field theory with massless bosons. These tools complement those developed in [4] and also their proof is to a large extent parallel to the latter. Therefore, we refer the reader to [4] for an extended motivation of the model and relevant references, and to [2] for a discussion of asymptotic completeness. Suffice it so say here that interest in the rigorous theory of such models was revived by work on non-relativistic quantum electrodynamics, see e.g. [1, 8] We first introduce the model and state the result, and then, in Section 1.4, we discuss the results in this paper.

1.1 The model

Our model consists of a small system (atom, spin) coupled to a free bosonic field. The Hilbert space of the total system is

$$\mathscr{H} = \mathscr{H}_{\mathrm{S}} \otimes \mathscr{H}_{\mathrm{F}} \tag{1.1}$$

where $\mathscr{H}_{\mathrm{S}}$, the atom/spin space (S for 'small system'), is finite dimensional, $\mathscr{H}_{\mathrm{S}} \sim \mathbb{C}^{d_{\mathrm{S}}}$ for some $d_{\mathrm{S}} < \infty$. The field space $\mathscr{H}_{\mathrm{F}}$ is the bosonic Fock space $\Gamma(\mathfrak{h})$ built from the single particle space $\mathfrak{h} = L^2(\mathbb{R}^d)$:

$$\mathscr{H}_{\mathrm{F}} = \Gamma(\mathfrak{h}) = \mathop{\oplus}_{n=0}^{\infty} P_{\mathrm{Symm}} \mathfrak{h}^{\otimes^n} \tag{1.2}$$

with P_{Symm} is the projection to symmetric tensors and $\mathfrak{h}^{\otimes^0} \equiv \mathbb{C}$. The total Hamiltonian is of the form

$$H = H_{\mathrm{S}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{F}} + H_{\mathrm{I}} \tag{1.3}$$

where

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1. H_{S} is a hermitian matrix acting on $\mathscr{H}_{\mathrm{S}}$.
2. H_{F} is the Hamiltonian of the free field, given by

$$H_{\mathrm{F}} = \int_{\mathbb{R}^d} \mathrm{d}k|k|a_k^* a_k \tag{1.4}$$

where a_k^*, a_k are the bosonic creation/annihilation operators of a Fourier (momentum) mode $k \in \mathbb{R}^d$ satisfying the 'Canonical Commutation Relations'

$$[a_k, a_{k'}^*] = \delta(k - k').$$

We also note that we have set the free propagation speed v_p (see abstract) to be 1 in choosing the 'dispersion law' to be $|k|$ rather than $v_p|k|$.

3. The coupling H_{I} is of the form

$$H_{\mathrm{I}} = \lambda D \otimes \Phi(\phi) \tag{1.5}$$

where D is a Hermitian matrix acting on $\mathscr{H}_{\mathrm{S}}$, $\lambda \in \mathbb{R}$ is a coupling constant, $\phi \in L^2(\mathbb{R}^d)$ is a "form factor" that imposes some infrared and ultraviolet regularity in the model and Φ is the self-adjoint (Segal) field operator

$$\Phi(\phi) = a^*(\phi) + a(\phi), \qquad a(\phi) = \int_{\mathbb{R}^d} \mathrm{d}k\, \overline{\hat{\phi}(k)} a_k \tag{1.6}$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ.

If the form factor ϕ satisfies

$$\int \mathrm{d}k|\hat{\phi}(k)|^2(1 + \frac{1}{|k|}) < \infty, \tag{1.7}$$

then the operator H_{I} is relatively bounded w.r.t. H_{F} with arbitrarily small relative bound and therefore the Hamiltonian H in (1.3) is self-adjoint on the domain of H_{F} by the Kato-Rellich theorem. Hence the unitary dynamics $\mathrm{e}^{-\mathrm{i}tH}$ is well-defined and we set $\Psi_t = \mathrm{e}^{-\mathrm{i}tH}\Psi_0$ with $\Psi_0 \in \mathscr{H}$. A lot of work has been devoted to this model, in particular to its spectral theory, but we do not discuss this here. Instead, references are collected in [4, 2].

1.2 Assumptions

We describe now our assumptions on the form factor. Its infrared (small Fourier mode k) behaviour determines temporal correlations in the model and some regularity near $k = 0$ is needed. Roughly speaking, we need to assume

$$\hat{\phi}(k) \sim |k|^{-\frac{d-2-\alpha}{2}} \tag{1.8}$$

with some $\alpha > 0$ as $|k| \to 0$.

Definition 1.1. Let $0 < \alpha < 1$. We define the subspace $\mathfrak{h}_\alpha \subset \mathfrak{h}$ to consist of $\psi \in \mathfrak{h}$ such that $\hat{\psi} \in C^3(\mathbb{R}^d \setminus \{0\})$, the support of $\hat{\psi}$ is bounded, and, for all multi indices m with $|m| \leq 3$,

$$|\partial_k^m \hat{\psi}(k)| \leq C|k|^{(\beta-d+2)/2-|m|}. \tag{1.9}$$

for some $\beta > \alpha$ and $C < \infty$.

In the following two assumptions, we fix once and for all the form factor, the dimension d, and the operators H_{S} and D. These choices and assumptions are assumed to hold throughout the article and they will not be repeated.

The first assumption controls the infrared behaviour of the model.

Assumption 1 (α-Infrared regularity)**.** *The form factor ϕ is in $\mathfrak{h}_\alpha$ and the dimension $d \geq 3$.*

One of the most intuitive consequences of this assumption is the decay of correlations for the free boson dynamics. Indeed, by stationary phase estimates, see Appendix, we get

$$\int \mathrm{d}k \, |\hat{\phi}(k)|^2 \mathrm{e}^{\mathrm{i}t|k|} = \mathcal{O}(t^{-(2+\alpha)}), \qquad t \to \infty. \tag{1.10}$$

The expression on the left hand side will appear in evaluating multitime correlations between the interaction terms H_I. The second assumption ensures that the coupling is effective. This assumption is very likely not necessary for our results, but it *is* required for our proof. To clarify this, consider the case $\phi = 0$, or equivalently $\lambda = 0$, then the atom and field are not coupled and the evolution of the field is given by (the quantisation of) the linear wave equation. In that case, our results can be proven by standard dispersive estimates. However, the proofs of the results in [4] (on which the present article is based) would break down as they rely on dissipative behaviour of the small system S, which does not occur if S is not coupled to the field.

Assumption 2 (Fermi Golden Rule)**.** *We assume that the spectrum of H_{S} is non-degenerate (all eigenvalues are simple) and we let $e_0 := \min \sigma(H_{\mathrm{S}})$ (atomic ground state energy). Most importantly, we assume that for any eigenvalue $e \in \sigma(H_{\mathrm{S}}), e \neq e_0$, there is a sequence $e(i), i = 1, \ldots, n$ of eigenvalues such that*

$$e = e(1) > e(2) > \ldots > e(n) = e_0, \qquad \textit{and} \qquad \forall i = 1, \ldots, n-1 : j(e(i), e(i+1)) > 0 \tag{1.11}$$

with $j(\cdot, \cdot)$ given by

$$j(e, e') := 2\pi \, \mathrm{Tr}[P_e D P_{e'} D P_e] \int_{\mathbb{R}^d} \mathrm{d}k \, \delta(|k| - (e - e')) |\hat{\phi}(k)|^2 \tag{1.12}$$

where P_e is the spectral projector corresponding to the eigenvalue e and the right hand side is well-defined since $\hat{\phi}$ is continuous away from 0.

This assumption will not enter explicitly in the present article, but it is necessary for the crucial Lemma 2.1, whose proof is in [4].

1.3 Results

We now state our main results. To choose appropriate initial states, we introduce the unitary *Weyl operator* $\mathcal{W}(\psi), \psi \in \mathfrak{h}$

$$\mathcal{W}(\psi) = \mathrm{e}^{\mathrm{i}\Phi(\psi)} \tag{1.13}$$

with the (Segal) field operator $\Phi(\psi)$ as in (1.6), and define the dense subspace

$$\mathcal{D}_\alpha := \mathrm{Span}\{\psi_{\mathrm{S}} \otimes \mathcal{W}(\psi)\Omega \,|\, \psi \in \mathfrak{h}_\alpha, \psi_{\mathrm{S}} \in \mathscr{H}_{\mathrm{S}}\} \tag{1.14}$$

with Ω the normalised vacuum vector. The density of $\mathcal{D}_\alpha$ in $\mathcal{H}$ follows from the density of $\mathfrak{h}_\alpha$ in $\mathfrak{h}$. We will choose the initial vector $\Psi_0 \in \mathcal{D}_\alpha$ with $\|\Psi_0\| = 1$ and we write $\Psi_t = \mathrm{e}^{-\mathrm{i}tH}\Psi_0$.

Fix a C^∞ function $\theta : \mathbb{R}^d \to [0, 1]$ with compact support in the ball centered at origin with radius $r_\theta < 1$. Since θ will be used to localise both in real $x-$space and in Fourier $k-$space we use the notation $\theta(x)$ for the multiplication operator and $\theta(k)$ for the Fourier multiplier. To any self-adjoint operator b on $\mathfrak{h}$, we associate its second quantisation $\mathrm{d}\Gamma(b)$, a self-adjoint operator on $\Gamma(\mathfrak{h})$, and we also write $\mathrm{d}\Gamma(b)$ for $\mathbb{1} \otimes \mathrm{d}\Gamma(b)$, acting on $\mathscr{H}$.

In the statement of our theorems, $\check{C}$ denotes constants that depend on Ψ_0, θ, α, the dimension d, d_{S}, and the parameters of the Hamiltonian (1.3), i.e. the form factor ϕ and the operators H_{S}, D, but not on λ. We recall that we always assume Assumptions 1 and 2 to hold.

Theorem 1.1 (Soft mode bound)**.** *There exists $\lambda_0 > 0$ such that for all λ with $0 < |\lambda| \leq \lambda_0$,*

$$\sup_{t \geq 0} |\langle \Psi_t, \mathrm{d}\Gamma(\theta(k/\delta))\Psi_t \rangle| \leq \check{C}\delta^{\alpha/2} \tag{1.15}$$

for any $\delta > 0$, smooth indicator θ as above, and $\Psi_0 \in \mathcal{D}_\alpha$.

This result complements the boson number bound in [4]:

$$\sup_{t \geq 0} |\langle \Psi_t, N\Psi_t \rangle| \leq \check{C} \tag{1.16}$$

with $N = \mathrm{d}\Gamma(\mathbb{1})$ the number operator. Although the infrared condition in that paper is slightly different, an obvious application of Lemma A.1 in Appendix A allows to derive that condition from our present infrared condition, i.e. from Assumption 1, such that (1.16) holds in the present framework as well. Inspecting the proof of Theorem 1.1, we see that the bound $\breve{C}\delta^{\alpha/2}$ in (1.15) can be replaced by $\breve{C}(\alpha')\delta^{\alpha'}$, for any $\alpha' < \alpha$, at the cost of making the constant dependent on α'.

Theorem 1.2 (Minimal velocity estimate)**.** *Let $0 < |\lambda| \leq \lambda_0$ as in Theorem 1.1 and fix an initial state vector $\Psi_0 \in \mathcal{D}_\alpha$ with $\|\Psi_0\| = 1$ and a smooth indicator θ as in Theorem 1.1. For any 'cutoff time' $t_c \geq |\lambda|^{-2}$, the limit*

$$a(t_c, \theta) := \lim_{t\to\infty} \langle \Psi_t, \mathrm{d}\Gamma(\theta(x/t_c))\Psi_t\rangle \tag{1.17}$$

exists and

$$|\langle \Psi_t, \mathrm{d}\Gamma(\theta(x/t_c))\Psi_t\rangle - a(t_c, \theta)| \leq \breve{C}(1+t)^{-\alpha}, \tag{1.18}$$

uniformly in t_c for $t \geq t_c \geq |\lambda|^{-2}$.

The restriction $t_c \geq |\lambda|^{-2}$ is not necessary for the result to hold, but its elimination requires an additional step in our proof, and therefore we avoided it, since we are mainly interested in the case $t_c = t$, see below. The obvious interpretation of Theorem 1.2 is that

$$a(t_c, \theta) = \langle \Psi_{\mathrm{gs}}, \mathrm{d}\Gamma(\theta(x/t_c))\Psi_{\mathrm{gs}}\rangle \tag{1.19}$$

where Ψ_{gs} is the unique (up to a phase) normalised ground state of H, that can indeed be proven to exist given our assumptions, see [4]. This interpretation is correct but we postpone its statement to [2] because the identification of the limit requires somehow different reasoning that does not naturally fit into the present paper. The most natural and, as far as we see, useful form of this result is obtained if we assume (1.19) and take $t_c = t$. Then the resulting statement is

$$|\langle \Psi_t, \mathrm{d}\Gamma(\theta(x/t))\Psi_t\rangle - \langle \Psi_{\mathrm{gs}}, \mathrm{d}\Gamma(\theta(x/t))\Psi_{\mathrm{gs}}\rangle| \leq \breve{C}(1+t)^{-\alpha} \tag{1.20}$$

which is a key ingredient in [2]. This is also the claim that was announced in the abstract.

1.4 Discussion

In [4], we established two results. On the one side, we showed that for localised observables O, i.e. those concerning the atom and the field in the neighbourhood of the atom, the expectation value $\langle \Psi_t, O\Psi_t\rangle$ converges to the stationary value $\langle \Psi_{\mathrm{gs}}, O\Psi_{\mathrm{gs}}\rangle$ (assuming that a ground state Ψ_{gs} exists). On the other side, we showed that the number of emitted bosons is bounded independently in time, i.e. (1.16). Intuition suggests that the emitted bosons behave as free bosons once they are sufficiently far from the atom. One consequence of this intuition is that the number of bosons in a spatial region of the form

$$c_1 t < |x| < c_2 t, \qquad \text{with } 0 < c_1 < c_2 < 1,$$

should tend to 0, as $t \to \infty$, where we recall that we have set the propagation speed of free bosons to be 1. In case $c_1 = 0$, this is not quite true since some bosons are bound by the atom in the interacting ground state, but in that case it is still true that the expectation value of the number of bosons tends to a constant value as $t \to \infty$. This result is achieved in Theorem 1.2 (up to some issues pointed out below the statement of this theorem). We refer to it as a minimal velocity estimate since it excludes the existence of bosons travelling with a speed lower than the propagation speed of free bosons.

Once one knows that the number of emitted bosons remains finite, such minimal velocity estimates can be obtained by operator techniques as well, but we prefer to modify slightly the polymer expansion in [4] to obtain these results. The approach via operator techniques has been explored in [6]. Both in our work, and in [6] the motivation comes from the fact that minimal velocity estimates are helpful in proving asymptotic completeness. However, in the present article, our aim is also to illustrate that the 'polymer expansion'-approach to problems in open quantum systems, that we started in [3], can be adapted to a variety of problems.

A second result in this paper is the soft mode bound, Theorem 1.1. This result could be obtained completely analogously to the treatment of [4], since the one-particle operator $b = \theta(k/\delta)$ is invariant under the free boson dynamics, but to make the present paper more streamlined, we treat it analogously to Theorem 1.2, which concerns a non-invariant b-operator. Note that in [5], such an analogy to [4] is used to control $\mathrm{d}\Gamma(b)$ for $b = 1/|k|$, which is of course also invariant.

1.5 Strategy of the proof and outline of the paper

Since the strategy of this paper is so intimately connected to [4], we restrict ourselves here to a rough outline of the proof. In particular, the arguments in Section 1.5.1 are analogous to [4] and they are more thoroughly explained there.

1.5.1 Polymer representation

We set out to control the quantity $\langle \Psi_t, \mathrm{d}\Gamma(b)\Psi_t\rangle$ for $t = n/\lambda^2$ and with the one-boson operator b being either $\theta(x/t_c)$ or $\theta(k/\delta)$. We first construct a *polymer representation* of this quantity:

$$\langle \Psi_t, \mathrm{d}\Gamma(b)\Psi_t\rangle = \sum_{\mathcal{A}} \sum_{A' \in \mathcal{A}} v(A' \cup \{n+1\}) \prod_{A \in \mathcal{A}, A \neq A'} v(A) \tag{1.21}$$

where A', A range over nonempty subsets of $\{0, 1, \dots, n\}$ (*polymers*) and $\mathcal{A}$ ranges over collections of polymers that are pairwise disjoint and non-adjacent. The initial state Ψ_0 influences the *polymer weights* $v(A)$ for $A \ni 0$ and the observable $\mathrm{d}\Gamma(b)$ influences $v(A)$ for $A \ni (n+1)$. For the weights of *bulk* polymers A, (i.e. not containing 0 nor $n+1$) we need a bound that was already stated in [4], and that, for small $|A|$, can be thought of as

$$|v(A)| \leq \lambda^2 (\max A - \min A)^{-(2+\alpha)} \tag{1.22}$$

where the decay factor $(\max A - \min A)^{-(2+\alpha)}$ reflects the decay of correlations of the free field exhibited in (1.10) and the factor λ^2 indicates that these polymers capture the effect of interactions.

If we replace $\mathrm{d}\Gamma(b)$ by $\mathbb{1}$, the corresponding expansion reads

$$1 = \langle \Psi_t, \mathbb{1}\Psi_t\rangle = \sum_{\mathcal{A}} \prod_{A \in \mathcal{A}} v(A), \tag{1.23}$$

i.e. polymers with $(n+1) \in A$ are absent. These polymer representations are derived in Sections 2.1 and 2.2 via an intermediate polymer representation with operator-valued polymer weights. This derivation is a purely algebraic exercise. Then we derive bounds on the polymer weights, like (1.22). We again use the operator-valued polymer weights as a useful intermediary step. This is done in Sections 2.3 and 2.4. The main idea of these bounds is to recognise in the definition of $v(A)$ a connected graph whose vertices, roughly speaking, coincide with the elements of A and whose edges $\{\tau, \tau'\}, \tau, \tau' \in A$ carry a decay factor $|\tau' - \tau|^{-(2+\alpha)}$. Since the graph is connected, we can extract the overall decay factor $(\max A - \min A)^{-(2+\alpha)}$. This description is oversimplified; in reality, some of the vertices of the graph are subsets of A themselves and the decay factors are encoded into them. The sum over graphs is performed with the help of combinatorial techniques from cluster expansions.

1.5.2 The weights $v(A \cup \{n+1\})$

Let us now describe the basic intuition for the weights $v(A \cup \{n+1\})$. First, by brutal approximation, we could guess that (1.21) divided by (1.23) is approximated as

$$\langle \Psi_t, \mathrm{d}\Gamma(b)\Psi_t\rangle \approx \sum_{A} v(A \cup \{n+1\}). \tag{1.24}$$

We will make this relation into an equality by replacing the weights v by slightly modified weights $\bar{v}$. This type of arguments are presented in Section 3.1 and they are again based on cluster expansions. Next, let us naively expand $\langle \Psi_t, \mathrm{d}\Gamma(b)\Psi_t\rangle$ in powers of λ, hence of H_I, up to second order. We write $H = H_0 + H_I$ and we use

$$\mathrm{e}^{\mathrm{i}tH_0} \mathrm{d}\Gamma(b) \mathrm{e}^{-\mathrm{i}tH_0} = \mathrm{e}^{\mathrm{i}tH_{\mathrm{F}}} \mathrm{d}\Gamma(b) \mathrm{e}^{-\mathrm{i}tH_{\mathrm{F}}} = \mathrm{d}\Gamma(b(t)) \tag{1.25}$$

where $b(t) = \mathrm{e}^{\mathrm{i}t\omega} b \mathrm{e}^{-\mathrm{i}t\omega}$ and ω is the self-adjoint multiplication operator with the Fourier multiplier $|k|$, i.e. $\widehat{(\omega\psi)}(k) = |k|\hat{\psi}(k)$. By (formal) Duhamel expansion of $\mathrm{e}^{\mathrm{i}tH}, \mathrm{e}^{-\mathrm{i}tH}$,

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}tH} \mathrm{d}\Gamma(b) \mathrm{e}^{-\mathrm{i}tH} + \mathcal{O}(|\lambda|^3) = \mathrm{d}\Gamma(b(t)) \\
&+ \mathrm{i} \int_0^t \mathrm{d}t_1\, H_I(t_1) \mathrm{d}\Gamma(b(t)) + \text{h.c.} \\
&+ \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\, H_I(t_1) \mathrm{d}\Gamma(b(t)) H_I(t_2) \\
&- \int_0^t \mathrm{d}t_1 \int_{t_1}^t \mathrm{d}t_2\, H_I(t_1) H_I(t_2) \mathrm{d}\Gamma(b(t)) + \text{h.c.}
\end{aligned}$$

where we write $H_I(t) = \mathrm{e}^{\mathrm{i}tH_0} H_I \mathrm{e}^{-\mathrm{i}tH_0}$ and (below) $D(t) = \mathrm{e}^{\mathrm{i}tH_0} D \mathrm{e}^{-\mathrm{i}tH_0}$. Let us now for simplicity choose $\Psi_0 = \psi_{\mathrm{S}} \otimes \Omega$ and put the above equation between $\langle \Psi_0, \cdot\, \Psi_0 \rangle$. Then all terms in the above expansion vanish except the third line, the integrand of which can be recast as

$$\langle \Psi_0, H_I(t_1) \mathrm{d}\Gamma(b(t)) H_I(t_2) \Psi_0 \rangle = \langle \psi_{\mathrm{S}}, D(t_1) D(t_2) \psi_{\mathrm{S}} \rangle_{\mathscr{H}_{\mathrm{S}}} \times \lambda^2 \langle a^*(\mathrm{e}^{\mathrm{i}t_1\omega}\phi)\Omega, \mathrm{d}\Gamma(b(t)) a^*(\mathrm{e}^{\mathrm{i}t_2\omega}\phi)\Omega \rangle.$$

The first factor on the left hand side is quasiperiodic in $t_2 - t_1$ and as such it is irrelevant. The second factor can be rewritten as

$$\lambda^2 \langle \mathrm{e}^{\mathrm{i}t_1\omega}\phi, b(t) \mathrm{e}^{\mathrm{i}t_2\omega}\phi \rangle, \tag{1.26}$$

i.e. an expression in the one-boson space $\mathfrak{h}$: we get the crude cartoon

$$\langle \Psi_t, \mathrm{d}\Gamma(b) \Psi_t \rangle \approx \lambda^2 \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\, \langle \mathrm{e}^{\mathrm{i}t_1\omega}\phi, b(t) \mathrm{e}^{\mathrm{i}t_2\omega}\phi \rangle. \tag{1.27}$$

Comparing this integral with the sum in (1.24), it is plausible that (1.26) is similar to

$$v(\{\tau_1, \tau_2, n+1\}), \qquad \text{with } \tau_1, \tau_2 \text{ such that } \tau_1 \approx \lambda^2 t_1 \text{ and } \tau_2 \approx \lambda^2 t_2, \tag{1.28}$$

which is not quite true, but good enough for the picture that we are developing here. The weights $v(A \cup \{n+1\})$ with $|A| > 2$ yield small corrections that we do not describe here.

One of the most relevant properties of the function (1.26) is that it retains the decay in $t_2 - t_1$ exhibited in (1.10). In the case $b = \theta(x/t_c)$, it is quite intricate to prove this uniformly in t_c, but this has little to do with our main technical work. Therefore, estimates of this kind are gathered in the Appendix, see in particular Lemma A.2.

1.5.3 The long-time limit

If one accepts the heuristic outline above, then one gets from (1.27) by the change of variables $s_i = t - t_i$, and abbreviating $M_b(s_1, s_2) := \lambda^2 \langle \mathrm{e}^{-\mathrm{i}s_1\omega}\phi, b\, \mathrm{e}^{-\mathrm{i}s_2\omega}\phi \rangle$

$$\lim_{t\to\infty} \langle \Psi_t, \mathrm{d}\Gamma(b) \Psi_t \rangle \approx \int\limits_{(\mathbb{R}_+)^2} \mathrm{d}s_1 \mathrm{d}s_2\, M_b(s_1, s_2), \tag{1.29}$$

$$\langle \Psi_t, \mathrm{d}\Gamma(b) \Psi_t \rangle - \lim_{t\to\infty} \langle \Psi_t, \mathrm{d}\Gamma(b) \Psi_t \rangle \approx - \int\limits_{(\mathbb{R}_+)^2 \setminus [0,t]^2} \mathrm{d}s_1 \mathrm{d}s_2\, M_b(s_1, s_2) \tag{1.30}$$

provided the improper integrals on the right hand side are defined by some appropriate regularisation procedure. If these were exact equalities, then our theorems would reduce to statements about one-boson dynamics. Those statements can then be checked by the estimates in the Appendix. Note that for $b = \theta(k/\delta)$, $M_b(s_1, s_2)$ is a function of $s_2 - s_1$ only, but its integral over $\mathbb{R}$ vanishes because of (1.8) and the above expressions are in fact finite. For $b = \theta(x/t_c)$, $M_b(s_1, s_2)$ decays as soon as s_1, s_2 are larger than t_c and also as $s_2 - s_1 \to \infty$; these are dispersive properties of the linear wave equation.

In Sections 3.3 and 3.4, we find the full-blown version of this argument, where the right hand sides of the above equations are replaced by sums over $\bar{v}(A \cup \{n+1\})$.

1.6 Notation

1.6.1 Combinatorics

We write $\mathbb{N} = \{0, 1, 2, \dots\}$. For $\tau, \tau' \in \mathbb{N}$, $\tau < \tau'$, we define the discrete intervals

$$I_{\tau,\tau'} := \{\tau, \tau+1, \dots, \tau'\} \tag{1.31}$$

and $\mathfrak{B}_{\tau,\tau'}$ the set of collections of nonempty subsets of $I_{\tau,\tau'}$. A relevant subset of $\mathfrak{B}_{\tau,\tau'}$ is, for $j \in \mathbb{N}$,

$$\mathfrak{B}^j_{\tau,\tau'} := \{\mathcal{A} \in \mathfrak{B}_{\tau,\tau'} \,|\, \forall A, A' \in \mathcal{A} : (A \neq A' \Rightarrow \mathrm{dist}(A, A') > j)\} \tag{1.32}$$

where $\mathrm{dist}(A, A') := \min_{\tau \in A, \tau' \in A'} |\tau - \tau'|$ (hence $\mathrm{dist}(A, A') = \infty$ if A or A' is empty). For a collection $\mathcal{A}$, we set

$$\mathrm{Supp}\mathcal{A} := \cup_{A \in \mathcal{A}} A \tag{1.33}$$

and we need also the diameter of (finite) subsets of $\mathbb{N}$;

$$d(A) = \max A - \min A + 1, \qquad d(\mathcal{A}) := d(\mathrm{Supp}\mathcal{A}). \tag{1.34}$$

1.6.2 Hilbert and Banach spaces

For a Banach space $\mathscr{E}$, we let $\mathscr{B}(\mathscr{E})$ stand for the set of bounded operators. If $\mathscr{E}$ is a Hilbert space, we will additionally use the space of trace class operators

$$\mathscr{B}_p(\mathscr{E}) = \{O \in \mathscr{B}(\mathscr{E}) \,|\, \|O\|_p < \infty\} \tag{1.35}$$

with

$$\|O\|_p = (\mathrm{Tr}\, |OO^*|^{p/2})^{1/p}. \tag{1.36}$$

For the scalar product on a Hilbert space $\mathscr{E}$, we use the notation $\langle \psi, \psi' \rangle_{\mathscr{E}}$, often abbreviated as $\langle \psi, \psi' \rangle$. A positive operator $\rho \in \mathscr{B}_1(\mathscr{E})$ with $\mathrm{Tr}\, \rho = 1$ is called a density matrix. We also use the function

$$\langle x \rangle := \sqrt{x^2 + 1}, \tag{1.37}$$

for real numbers and self-adjoint operators.

1.6.3 Constants

We denote by c, C constants that depend only on the dimensions d, d_{S}, the parameters of the Hamiltonian 1.3 and the parameter α, but not on λ. The precise value of these constants can be different in different equations. Quantities that additionally depend on the initial condition Ψ_0 and the smooth function θ (but not on λ) are denoted by $\breve{c}, \breve{C}$.

2 Polymer Representation

In this section, we complete the first important step of our proof, namely we rewrite all quantities of interest through a polymer representation. This part of the paper is almost identical to a corresponding part in [4].

We discretise time by introducing a "mesoscopic" time scale λ^{-2}. That is, we consider times of the form $t = n/\lambda^2$ with $n \in \mathbb{N}$. The discretisation will be removed at the end of the argument. We study

$$Z_n(O, \rho_0) := \mathrm{Tr}\left[O \mathrm{e}^{-\mathrm{i}tH} \rho_0 \mathrm{e}^{\mathrm{i}tH}\right], \qquad t = n/\lambda^2, \tag{2.1}$$

with the initial density matrix

$$\rho_0 = \rho_{\mathrm{S},0} \otimes \mathcal{W}(\psi_{\ltimes}) P_\Omega \mathcal{W}^*(\psi_{\ltimes})$$

for some density matrix $\rho_{\mathrm{S},0} \in \mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$, $\psi_{\ltimes} \in \mathfrak{h}_\alpha$, and P_Ω the one-dimensional projector on the range of the vacuum vector Ω. The observable O is one of the following

1. $O = \mathrm{d}\Gamma(b)$ with $b = b_x := \theta(x/t_c)$ or $b = b_k := \theta(k/\delta)$. We choose $t_c = \lambda^{-2} n_c$ for some $n_c \in \mathbb{N}, n_c \leq n$. All estimates will be uniform in n_c. Since O is unbounded, the expression (2.1) is in need of justification, which is provided in [4]. A posteriori, one can also appeal to the convergent expansions developed in Section 2.3 where we construct a convergent expansion for (2.1).

2. $O = \mathbb{1}$. This case is mainly included for comparison. By cyclicity of the trace, we have $Z_n(\mathbb{1}, \rho_0) = 1$.

In most intermediary steps of our analysis we will perform a partial trace over the field, thereby defining the reduced dynamics

$$Q_n \rho_{\mathrm{S},0} := \mathrm{Tr}_{\mathrm{F}} \left[\mathrm{e}^{-\mathrm{i}(n/\lambda^2)L} \rho_0 \right] \tag{2.2}$$

where we introduced the Liouvillian $L = \mathrm{ad}(H)$, an unbounded operator on $\mathscr{B}_1(\mathscr{H})$. Sometimes, we want to incorporate the observable into the reduced analysis as well. In that case, we write

$$Q_{n|b} \rho_{\mathrm{S},0} := \mathrm{Tr}_{\mathrm{F}} \left[\mathrm{d}\Gamma(b) \mathrm{e}^{-\mathrm{i}(n/\lambda^2)L} (\rho_{\mathrm{S},0} \otimes P_\Omega) \right]. \tag{2.3}$$

Here the notation differs slightly from the one in [4] where the latter object was called $\check{Q}_n$ and the notation Q_n was reserved for (2.2) with $\psi_\ltimes = 0$. Obviously, we have

$$Z_n(\mathbb{1}, \rho_0) = \mathrm{Tr}\, Q_n \rho_{\mathrm{S},0} = 1, \qquad Z_n(\mathrm{d}\Gamma(b), \rho_0) = \mathrm{Tr}\, Q_{n|b} \rho_{\mathrm{S},0}. \tag{2.4}$$

The main goal of the first part of the present chapter is to find a convenient representation for Q_n and $Q_{n|b}$. The first step is to write the evolution operators as a product where each factor corresponds to a 'mesoscopic' time slice of length λ^{-2}. With this in mind, we introduce

$$U_\tau : \mathscr{B}_1(\mathscr{H}) \to \mathscr{B}_1(\mathscr{H}) \tag{2.5}$$

with

$$U_\tau := \mathrm{e}^{\mathrm{i}(\tau/\lambda^2)L_{\mathrm{F}}} \mathrm{e}^{-\mathrm{i}(1/\lambda^2)L} \mathrm{e}^{-\mathrm{i}((\tau-1)/\lambda^2)L_{\mathrm{F}}}, \qquad \tau \in I_{1,n}, \tag{2.6}$$

where we used the field Liouvillian $L_{\mathrm{F}} := \mathrm{ad}(H_{\mathrm{F}})$, and

$$U_0 \rho := \mathcal{W}(\psi_\ltimes) \rho \mathcal{W}^*(\psi_\ltimes), \tag{2.7}$$

$$U_{n+1} \rho := \mathrm{d}\Gamma(b(n/\lambda^2)) \rho, \tag{2.8}$$

where we wrote for brevity $\mathcal{W}(\psi)$ instead of $\mathbb{1} \otimes \mathcal{W}(\psi)$ and $b(t) = \mathrm{e}^{\mathrm{i}t\omega} b \mathrm{e}^{-\mathrm{i}t\omega}$ An immediate consequence of these definitions, using cyclicity of the trace, is

$$Q_n \rho_{\mathrm{S},0} := \mathrm{Tr}_{\mathrm{F}} \left[U_n \ldots U_1 U_0 (\rho_{\mathrm{S},0} \otimes P_\Omega) \right], \tag{2.9}$$

$$Q_{n|b} \rho_{\mathrm{S},0} := \mathrm{Tr}_{\mathrm{F}} \left[U_{n+1} U_n \ldots U_1 U_0 (\rho_{\mathrm{S},0} \otimes P_\Omega) \right]. \tag{2.10}$$

Finally, we define the reduced dynamics

$$T : \mathscr{B}_1(\mathscr{H}_{\mathrm{S}}) \to \mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$$

for (mesoscopic) time 1, starting from a product state

$$T \rho_{\mathrm{S},0} := \mathrm{Tr}_{\mathrm{F}} \left[\mathrm{e}^{-\mathrm{i}(1/\lambda^2)L} (\rho_{\mathrm{S},0} \otimes P_\Omega) \right]. \tag{2.11}$$

We set $T_\tau := T$ for $\tau = 1, \ldots, n$ and

$$T_0 \;:=\; \langle \mathcal{W}(\psi_\ltimes)\Omega, \mathcal{W}(\psi_\ltimes)\Omega \rangle \, \mathbb{1} = \mathbb{1}, \tag{2.12}$$

$$T_{n+1} \;:=\; \langle \Omega, \mathrm{d}\Gamma(b)\Omega \rangle \, \mathbb{1} = 0 \tag{2.13}$$

where we used that $\|\mathcal{W}(\psi_\ltimes)\Omega\| = 1$ because $\mathcal{W}(\psi_\ltimes)$ is unitary. The motivation for this definition will become obvious in the next section. Finally, we set

$$B_\tau := U_\tau - T_\tau, \qquad \tau = 0, \ldots, n+1. \tag{2.14}$$

Note that U_τ depends on the total macroscopic time n because of (2.8). It is sometimes convenient, see e.g. Section 2.1.2, to indicate the n dependence explicitly by writing $U_{\tau,n}, B_{\tau,n}$ with $\tau \leq n+1$, such that $U_{n+1,n}$ is defined by (2.8) and $U_{\tau,n}, \tau \leq n$ is defined by (2.6,2.7).

The next section proposes a framework whose purpose is to write an expansion for Q_n and $Q_{n|b}$ in which the leading terms, in a precise sense, are $T_n \ldots T_2 T_1 T_0 = T^n T_0$ and $T_{n+1} T^n T_0 = 0$, respectively.

2.1 Operator-valued polymers weights

2.1.1 Operator correlation functions

We abbreviate

$$\mathscr{R}_{\mathrm{S}}=\mathscr{B}(\mathscr{B}_1(\mathscr{H}_S)), \qquad \mathscr{R}_{\mathrm{F}}=\mathscr{B}(\mathscr{B}_1(\mathscr{H}_{\mathrm{F}})). \tag{2.15}$$

Define, for $W,W'\in\mathscr{R}_{\mathrm{S}}\otimes\mathscr{R}_{\mathrm{F}}$ the object

$$W\otimes_{\mathrm{S}}W'\in\mathscr{R}_{\mathrm{S}}\otimes\mathscr{R}_{\mathrm{S}}\otimes\mathscr{R}_{\mathrm{F}}$$

as an operator product in F-part and tensor product in S-part. Concretely, let $W=W_{\mathrm{S}}\otimes W_{\mathrm{F}}$ and $W'=W'_{\mathrm{S}}\otimes W'_{\mathrm{F}}$. Then

$$W\otimes_{\mathrm{S}}W':=W_{\mathrm{S}}\otimes W'_{\mathrm{S}}\otimes W_{\mathrm{F}}W'_{\mathrm{F}}.$$

and we extend this by linearity to arbitrary W,W'. Iterating this construction we define for $W_i\in\mathscr{R}_{\mathrm{S}}\otimes\mathscr{R}_{\mathrm{F}}$, $i=1,\ldots,m$

$$W_m\otimes_{\mathrm{S}}\ldots\ldots\otimes_{\mathrm{S}}W_2\otimes_{\mathrm{S}}W_1\in(\mathscr{R}_{\mathcal{S}})^{\otimes^m}\otimes\mathscr{R}_{\mathrm{F}}.$$

Since $\mathscr{R}_{\mathrm{S}}$ is finite-dimensional these products are unambiguously defined.

We define the 'expectation'

$$\mathbb{E}:(\mathscr{R}_{\mathrm{S}})^{\otimes^m}\otimes\mathscr{R}_{\mathrm{F}}\ \to\ (\mathscr{R}_{\mathrm{S}})^{\otimes^m}$$

as

$$\mathbb{E}(W)J:=\mathrm{Tr}_{\mathrm{F}}[W(J\otimes P_\Omega)], \qquad J\in(\mathscr{B}_1(\mathscr{H}_{\mathrm{S}}))^{\otimes^m}.$$

Obviously, the action of $\mathbb{E}$ is extended to unbounded W satisfying $W((\mathscr{B}_1(\mathscr{H}_{\mathrm{S}}))^{\otimes^m}\otimes P_\Omega)\in\mathscr{B}_1(\mathscr{H}_{\mathrm{S}}^{\otimes^m}\otimes\mathscr{H}_{\mathrm{F}})$. An important example, with $m=1$, is $T=\mathbb{E}(U_\tau)$.

Let $A=\{\tau_1,\tau_2,\ldots,\tau_m\}\subset I_{0,n+1}$ with the convention that $\tau_i<\tau_{i+1}$ and define the 'time-ordered correlation function'

$$G_A:=\mathbb{E}\left(B_{\tau_m}\otimes_{\mathrm{S}}B_{\tau_{n-1}}\otimes_{\mathrm{S}}\cdots\otimes_{\mathrm{S}}B_{\tau_1}\right)\in(\mathscr{R}_S)^{\otimes^m}. \tag{2.16}$$

Note that $G_A=0$ when the set A is a singleton. Indeed, since $B_\tau=U_\tau-\mathbb{E}(U_\tau)$. we get $\mathbb{E}(B_\tau)=0$.

It will be convenient to label the $\mathscr{R}_{\mathrm{S}}$'s and to drop the subscript S (since we will rarely need $\mathscr{R}_{\mathrm{F}}$), writing simply $\mathscr{R}$ for $\mathscr{R}_{\mathrm{S}}$. Let us denote by $\mathscr{R}^{\otimes^{\mathbb{N}}}$ the linear space spanned by simple tensors $\ldots\otimes V_2\otimes V_1$ where all but a finite number of V_j are equal to the identity $\mathbb{1}$. For finite subsets $A\subset\mathbb{N}$, we then define $\mathscr{R}_A$ as the finite-dimensional subspace of $\mathscr{R}^{\otimes^{\mathbb{N}}}$ spanned by simple tensors $\ldots\otimes V_2\otimes V_1$ with $V_j=\mathbb{1}, j\notin A$ and we write in particular $\mathscr{R}_\tau=\mathscr{R}_{\{\tau\}}$. Let $A=\{\tau_1,\tau_2,\ldots,\tau_m\}$ with $\tau_1<\tau_2<\ldots<\tau_m$. Obviously, $\mathscr{R}_A$ is naturally isomorphic to $\mathscr{R}^{\otimes^m}$ by identifying the right-most tensor factor to $\mathscr{R}_{\tau_1}$, the next one to $\mathscr{R}_{\tau_2}$, etc... We denote this isomorphism from $\mathscr{R}^{\otimes^m}$ to $\mathscr{R}_A$ by $\boldsymbol{I}_A$ and we will from now on write G_A to denote $\boldsymbol{I}_A[G_A]\in\mathscr{R}_A$ since G_A acting on the 'unlabelled' space $\mathscr{R}^{\otimes^m}$ will not be used, except briefly in the upcoming Section 2.1.2

Consider a collection $\mathcal{A}$ of disjoint subsets of $\mathbb{N}$, then each of the spaces $\mathscr{R}_{A\in\mathcal{A}}$ is a subspace of $\mathscr{R}_{\mathrm{Supp}\mathcal{A}}$ where $\mathrm{Supp}\mathcal{A}=\cup_{A\in\mathcal{A}}A$. Given a collection of operators $K_A\in\mathscr{R}_A$, we have $\prod_A K_A\in\mathscr{R}_{\mathrm{Supp}\mathcal{A}}$. However, we prefer to denote such products by

$$\mathop{\otimes}_{A\in\mathcal{A}}K_A\in\mathscr{R}_{\mathrm{Supp}\mathcal{A}}, \tag{2.17}$$

i.e. we keep the tensor product explicit in the notation.

2.1.2 Symmetry properties

For later use, we also establish some symmetry properties of the operators G_A. To do this, it is more natural to keep the definition (2.16), i.e. to view G_A as an element of $\mathscr{R}^{\otimes^m}$ with $m=|A|$ instead of $\mathscr{R}_A$. As announced following equation (2.14), we write $B_{\tau,n}$ instead of B_τ and then also $G_{A,n}$ instead of G_A to indicate the dependence on the final time n:

Let $\tau\in\mathbb{N}$, $n'>n$ and A such that both $A,A+\tau$ are subsets of $I_{1,n}$, then

$$G_{A+\tau,n}=G_{A,n}=G_{A,n'} \tag{2.18}$$

because $\mathrm{e}^{-\mathrm{i}tL_{\mathrm{F}}}P_\Omega=P_\Omega$. Similarly, if $b=b_k$, then, with A,τ,n' as above

$$G_{(A+\tau)\cup\{n+1\},n}=G_{A\cup\{n+1\},n}=G_{A\cup\{n'+1\},n'}. \tag{2.19}$$

This follows by the invariance of the observable under free dynamics: $\mathrm{e}^{\mathrm{i}tL_{\mathrm{F}}}\mathrm{d}\Gamma(b_k) = \mathrm{d}\Gamma(b_k(t)) = \mathrm{d}\Gamma(b_k)$. Finally, if $b = b_x$, then the above properties do not hold in general, but we still have invariance under joint translations of A and the final time n, i.e.

$$G_{A\cup\{n+1\},n} = G_{(A+\tau)\cup\{n+\tau+1\},n+\tau} \tag{2.20}$$

from $\mathrm{e}^{-\mathrm{i}tL_{\mathrm{F}}}P_\Omega = P_\Omega$.

2.1.3 The contraction operator $\mathcal{T}$

We define the "contraction operator" $\mathcal{T} : \mathscr{R}_A \to \mathscr{R}$, by first giving its action on elementary tensors: Consider a family of operators $V_\tau \in \mathscr{R}$, and set

$$\mathcal{T}\left[\underset{\tau\in A}{\otimes} \boldsymbol{I}_\tau[V_\tau]\right] = V_{\tau_m}V_{\tau_{m-1}}\dots V_{\tau_1}, \qquad \text{where} \qquad \tau_m > \tau_{m-1} > \dots > \tau_1, \tag{2.21}$$

and then extend linearly to the whole of $\mathscr{R}_A$. On the left hand side, we will from now on abbreviate $\boldsymbol{I}_\tau[V_\tau]$ by V_τ. This is a slight abuse of notation that should not cause confusion because we keep the tensor products explicit in the notation, as explained in Section 2.1.1.

By expanding $U_\tau = T\otimes \mathbb{1} + B_\tau$ for every τ in the expression for the reduced dynamics (2.9), we arrive at

$$Q_n = \sum_{A\subset I_{0,n}} \mathcal{T}\left[G_A \underset{\tau\in I_{0,n}\setminus A}{\otimes} T_\tau\right] \tag{2.22}$$

where, for $A = \emptyset$, we mean to omit G_A from the right hand side. Similarly, for (2.10) we get

$$Q_{n|b} = \sum_{A\subset I_{0,n+1}} \mathcal{T}\left[G_A \underset{\tau\in I_{0,n+1}\setminus A}{\otimes} T_\tau\right]. \tag{2.23}$$

It is clear that, in the latter formula, only A with $n+1 \in A$ give a non-zero contribution because $T_{n+1} = 0$.

2.1.4 Connected correlations

Analogously to classical probability, we define the *connected correlation functions* or cumulants $G^c_A \in \mathscr{R}_A$ for nonempty A, satisfying

$$G_A = \sum_{\mathcal{A}} \underset{A\in\mathcal{A}}{\otimes} G^c_A$$

where $\mathcal{A}$ run through the set of partitions of A. As in the classical setup, G^c_A can be solved from this inductively in $|A|$, i.e.

$$G^c_\tau = G_\tau, \qquad G^c_{\{\tau_1,\tau_2\}} = G_{\{\tau_1,\tau_2\}} - G^c_{\tau_2}\otimes G^c_{\tau_1}, \tag{2.24}$$

$$G^c_{\{\tau_1,\tau_2,\tau_3\}} = G_{\{\tau_1,\tau_2,\tau_3\}} - \sum_{j=1,2,3} G^c_{\tau_j}\otimes G^c_{\{\tau_1,\tau_2,\tau_3\}\setminus\{\tau_j\}} - \underset{j=1,2,3}{\otimes} G^c_{\tau_j}. \tag{2.25}$$

Note that it is a consequence and advantage of our conventions that the order in which we write the tensors does not matter. We obtain then

$$Q_n = \sum_{\mathcal{A}\in\mathfrak{B}^0_{0,n}} \mathcal{T}\left[\underset{A\in\mathcal{A}}{\otimes} G^c_A \underset{\tau\in I_{0,n}\setminus\mathrm{Supp}\mathcal{A}}{\otimes} T_\tau\right], \tag{2.26}$$

$$Q_{n|b} = \sum_{\mathcal{A}\in\mathfrak{B}^0_{0,n+1}} \mathcal{T}\left[\underset{A\in\mathcal{A}}{\otimes} G^c_A \underset{\tau\in I_{0,n+1}\setminus\mathrm{Supp}\mathcal{A}}{\otimes} T_\tau\right]. \tag{2.27}$$

It is immediately clear that any contribution to the sum in (2.27) vanishes unless $n+1 \in \mathrm{Supp}\mathcal{A}$ because of $T_{n+1} = 0$.

2.1.5 Norms

Let us introduce a convenient norm on the spaces $\mathscr{R}_A$. For $E \in \mathscr{R}$ (i.e. the case $|A| = 1$), we set

$$\|E\|_\diamond := \|E\| = \sup_{\rho \in \mathscr{B}_1(\mathscr{H}_\mathrm{S}), \|\rho\|_1 = 1} \|E(\rho)\|_1, \tag{2.28}$$

i.e. the natural operator norm on $\mathscr{B}(\mathscr{B}_1(\mathscr{H}_\mathrm{S}))$.

For $E \in \mathscr{R}_A$ with $1 < |A| < \infty$, we exploit that E can be written as a finite sum of elementary tensors

$$E = \sum_\nu E_\nu, \qquad E_\nu = \otimes_{\tau \in A} E_{\nu,\tau}, \qquad E_{\nu,\tau} \in \mathscr{R}_\tau, \tag{2.29}$$

to define

$$\|E\|_\diamond := \inf_{\{E_\nu\}} \sum_\nu \prod_{\tau \in A} \|E_{\nu,\tau}\| \tag{2.30}$$

where the infimum ranges over all such elementary tensor-representations of E. This norm is useful because of the following properties (trivial from the definition):

1) For any family of operators $K_{A \in \mathcal{A}}$ with $K_A \in \mathscr{R}_A$ and $\mathcal{A}$ a collection of disjoint sets, we have

$$\left\| \underset{A \in \mathcal{A}}{\otimes} K_A \right\|_\diamond \leq \prod_{A \in \mathcal{A}} \|K_A\|_\diamond. \tag{2.31}$$

2) For any $K_A \in \mathscr{R}_A$,

$$\|\mathcal{T}[K_A]\| \leq \|K_A\|_\diamond. \tag{2.32}$$

2.2 Scalar polymer weights

The representations (2.26, 2.27) evoke the picture of a leading dynamics T interrupted by excitations, indexed by the sets $A \in \mathcal{A}$, and with operator valued weights G^c_A. We will now construct a similar representation, but with scalar weights. We exploit the dissipativity of the model, captured in the upcoming lemma. For operators $W' \in \mathscr{B}_1(\mathscr{H}_\mathrm{S}), W \in \mathscr{B}(\mathscr{H}_\mathrm{S})$, we write $|W'\rangle\langle W|$ to denote the operator in $\mathscr{R}$ acting as $S \mapsto |W'\rangle\langle W|S = W' \operatorname{Tr}(W^* S)$

Lemma 2.1. *Recall the operator $T \in \mathscr{R}$ defined in* (2.11)*. It has a simple eigenvalue equal to* 1*, corresponding to the one-dimensional spectral projector $R = |\eta\rangle\langle \mathbb{1}|$, with η a density matrix, such that*

$$\|T^m - R\| \leq C\mathrm{e}^{-gm} \tag{2.33}$$

for some $g > 0$.

This is Lemma 2.3 1) in [4] specialised to the case $\kappa = 0$. We exploit this to split

$$T = R + T^\perp, \tag{2.34}$$

where $T^\perp := T - R$ and we have

$$RT^\perp = T^\perp R = 0, \qquad TR = RT = R. \tag{2.35}$$

Analogously, we define $T_0 = TR + T_0^\perp = R + (\mathbb{1} - R)$ (since $T_0 = \mathbb{1}$) so that (2.35) also holds for T_0. We will insert these decompositions into the expansions (2.26, 2.27). The following definition provides the tools for this

Definition 2.1 (Fusions)**.** Let $\mathcal{A} \in \mathfrak{B}^0_{0,n}$ and let $\mathcal{J} \in \mathfrak{B}^1_{0,n}$ with the property that all $J \in \mathcal{J}$ are intervals. We say that a pair $(\mathcal{A}, \mathcal{J})$ is a *fusion* if

1. $\mathrm{Supp}\mathcal{A} \cap \mathrm{Supp}\mathcal{J} = \emptyset$.
2. $\mathrm{dist}(I_{0,n} \setminus \mathrm{Supp}(\mathcal{A} \cup \mathcal{J}), \mathrm{Supp}\mathcal{J}) > 1$.
3. The following undirected graph $\Gamma(\mathcal{A}, \mathcal{J})$ is connected. Its vertex set is the disjoint union $\mathcal{A} \sqcup \mathcal{J}$, and its edges are $\{A, J\}$ with $A \in \mathcal{A}, J \in \mathcal{J}, \mathrm{dist}(A, J) = 1$ and $\{A, A'\}$ with $A, A' \in \mathcal{A}, \mathrm{dist}(A, A') = 1$.

The set of fusions is denoted by $\mathfrak{S}^f_n$.

Remark 2.2. *The only fusions* $(\mathcal{A},\mathcal{J})$ *with* $\mathcal{A}=\emptyset$ *are* $(\emptyset,\emptyset)$ *and* $(\emptyset,\{I_{0,n}\})$. *The fusion* $(\emptyset,\emptyset)$ *will not play any role in what follows because its support,* $\mathrm{Supp}(\mathcal{A}\cup\mathcal{J})$*, is empty.*

Define now, for a fusion $(\mathcal{A},\mathcal{J})$,

$$V((\mathcal{A},\mathcal{J})) := \underset{A\in\mathcal{A}}{\otimes} G^c_A \underset{\tau\in\mathrm{Supp}\mathcal{J}}{\otimes} T^{\perp}_{\tau} \tag{2.36}$$

as an operator in $\mathscr{R}_{\mathrm{Supp}(\mathcal{A}\cup\mathcal{J})}$. By summing fusions with the same support, we set

$$\Sigma V(A) := \sum_{(\mathcal{A},\mathcal{J})\in\mathfrak{S}^f_n:\mathrm{Supp}(\mathcal{A}\cup\mathcal{J})=A} V((\mathcal{A},\mathcal{J})). \tag{2.37}$$

We can now regroup terms in (2.26) such that

$$Q_n = \sum_{\mathcal{A}\in\mathfrak{B}^1_{0,n}} \mathcal{T}\left[\underset{\tau\in(\mathrm{Supp}\mathcal{A})^c}{\otimes} R_\tau \underset{A\in\mathcal{A}}{\otimes} \Sigma V(A)\right] \tag{2.38}$$

where $(\mathrm{Supp}\mathcal{A})^c = I_{0,n}\setminus\mathrm{Supp}\mathcal{A}$. We refer the reader to [4] for a step by step derivation of this formula, that starts by splitting $T_\tau = R_\tau + T^{\perp}_\tau$ in (2.26).

Note that since $\mathcal{A}\in\mathfrak{B}^1_{0,n}$ the sets $A\in\mathcal{A}$ above are non-adjacent, i.e. distances between them are greater than 1. Hence, for any $\mathcal{A}$ in the formula above, all τ that are adjacent to the set $\mathrm{Supp}\mathcal{A}$ carry the rank-one operator R. A pictorial way to phrase this is that any of the operators $\Sigma V(A)$ in (2.38) is surrounded by projections R, except possibly at the boundaries of the interval $I_{0,n}$. We exploit this by defining, for $A\neq\emptyset$,

$$\hat{v}(A) := \mathcal{T}\left[\Sigma V(A) \bigotimes_{\tau\in I_{0,n}\setminus A} R_\tau\right], \qquad \hat{v}(A)\in\mathscr{R}. \tag{2.39}$$

Note that $\hat{v}(A)$ is a multiple of R unless $0\in A$ and/or $n\in A$. Finally, we recall that $R=|\eta\rangle\langle\mathbb{1}|$ and define

$$v(A) := \begin{cases} \langle\mathbb{1},\hat{v}(A)\eta\rangle & 0\notin A \\ \langle\mathbb{1},\hat{v}(A)\rho_{\mathrm{S},0}\rangle & 0\in A \end{cases} \tag{2.40}$$

With these definitions, one can check that we obtain

$$Z_n(\mathbb{1},\rho_0) = \mathrm{Tr}\, Q_n\rho_{\mathrm{S},0} = \sum_{\mathcal{A}\in\mathfrak{B}^1_{0,n}} \prod_{A\in\mathcal{A}} v(A) \tag{2.41}$$

where we have used the fact that $\mathrm{Tr}\,\rho_{\mathrm{S},0} = \langle\mathbb{1},\rho_{\mathrm{S},0}\rangle = 1$ to simplify the formula, and the summand on the right hand side is understood to be 1 for $\mathcal{A}=\emptyset$. Again, a more detailed derivation can be found in [4] (compared to the corresponding expression in [4] the factors $k_{\ltimes}k_{\rtimes}$ are missing, $k_{\ltimes}$ is missing because $\mathrm{Tr}\,\rho_{\mathrm{S},0}=1$ and $k_{\rtimes}$ is missing because, unlike in [4], we don't have an observable consisting of Weyl-operators). In the special case where $\rho_0=\eta\otimes P_\Omega$, using $\mathrm{Tr}\,\rho_0=1$, (2.41) reduces to

$$1 = Z_n(\mathbb{1},\eta\otimes P_\Omega) = \sum_{\mathcal{A}\in\mathfrak{B}^1_{1,n}} \prod_{A\in\mathcal{A}} v(A) \tag{2.42}$$

because in that case $v(A)=0$ whenever $0\in A$. This follows from $\psi_{\ltimes}=0$ and $T^{\perp}\eta=0$.

Remark 2.3. *Fusions* $(\mathcal{A},\mathcal{J})$ *with* $n\in\mathrm{Supp}\mathcal{J}$ *do not contribute to* $v(\cdot)$. *Indeed, they contribute to* $\hat{v}(\cdot)$ *an operator of the form* $T^{\perp}K$ *for some* $K\in\mathscr{R}$*, but we have*

$$\mathrm{Tr}(T^{\perp}K\rho) = \mathrm{Tr}\, TK\rho - \mathrm{Tr}\, RK\rho = 0$$

because T *and* R *conserve the trace. In particular, by Remark 2.2, fusions with* $\mathcal{A}=\emptyset$ *do not contribute.*

It remains to generalise this formula to the case where we have the observable $\mathrm{d}\Gamma(b)$. As already indicated, this is taken care of by defining the boundary element $n+1$. One could generalise the concepts above, like fusions, to include this element in an appropriate way, but we prefer not to do this, the reason being that the boundary element $n+1$ behaves in a very distinct way. Instead, we proceed as follows: Fix a fusion $(\mathcal{A},\mathcal{J})$ with $\mathcal{A}\neq\emptyset$ and a set $A\in\mathcal{A}$. We modify the collection $\mathcal{A}$ by replacing the set A by $A\cup\{n+1\}$ and calling the obtained collection $\mathcal{A}_A$, i.e.

$$\mathcal{A}_A := (\mathcal{A}\setminus\{A\})\cup\{A\cup\{n+1\}\}. \tag{2.43}$$

We can then define the operator $V((\mathcal{A}_A,\mathcal{J}))$ via (2.36) as an operator on $\mathscr{R}_{\mathrm{Supp}(\mathcal{A}\cup\mathcal{J})\cup\{n+1\}}$ because G^c_A with $n+1\in A$ is well-defined. Then we set

$$\Sigma V(A'\cup\{n+1\}) := \sum_{\substack{(\mathcal{A},\mathcal{J})\in\mathfrak{G}^f_n,\mathcal{A}\neq\emptyset\\ \mathrm{Supp}(\mathcal{A}\cup\mathcal{J})=A'}}\ \sum_{A\in\mathcal{A}} V((\mathcal{A}_A,\mathcal{J})), \tag{2.44}$$

and we simply define $\hat{v}(A\cup\{n+1\})$ and $v(A\cup\{n+1\})$ by the relations (2.39) and (2.40) with A replaced by $A\cup\{n+1\}$. For consistency with later formulas, we also set $v(\{n+1\})=0$. Note that we do not extend the setup to include the possibility that $n+1\in\mathrm{Supp}\mathcal{J}$. This is indeed not necessary since such a contribution would necessarily vanish because $T_{n+1}=0$, see (2.13). Now, the final expression for $Z_n(\mathrm{d}\Gamma(b),\rho_0)$ reads

$$Z_n(\mathrm{d}\Gamma(b),\rho_0) = \sum_{\mathcal{A}\in\mathfrak{B}^1_{0,n}}\sum_{A\in\mathcal{A}} v(A\cup\{n+1\}) \prod_{A'\in\mathcal{A}\setminus\{A\}} v(A') \tag{2.45}$$

where it is understood that $\mathcal{A}=\emptyset$ does not contribute to the right hand side and the empty product is set to 1.

2.3 Estimates on operator-valued polymers

2.3.1 Dyson expansion

We will now derive a formula for the correlation functions G^c_A in graphical terms. Recalling that $H=H_S+H_F+H_I$ we decompose $L=\mathrm{ad}(H)$ as

$$L = L_\mathrm{F}+L_\mathrm{S}+L_\mathrm{I} \tag{2.46}$$

and introduce

$$L_\mathrm{I}(s) = \begin{cases} \mathrm{e}^{\mathrm{i}sL_\mathrm{F}}L_\mathrm{I}\mathrm{e}^{-\mathrm{i}sL_\mathrm{F}} & s\geq 0\\ \mathrm{ad}(\Phi(\psi_\ltimes)) & -1\leq s<0\end{cases} \tag{2.47}$$

We develop the evolution operator $\mathrm{e}^{-\mathrm{i}tL}$ and the Weyl operator $\mathcal{W}(\psi_\ltimes)$ in a standard way in a Dyson expansion, arriving at

$$\mathrm{e}^{\mathrm{i}tL_\mathrm{S}}Q_n\rho_{\mathrm{S},0} = \sum_{m\in\mathbb{N}}(-1)^m \int\limits_{-1\leq t_1<\ldots<t_{2m}<n/\lambda^2} \mathrm{d}t_1\ldots\mathrm{d}t_{2m}\ \mathrm{Tr}_\mathrm{F}\left[L_\mathrm{I}(t_{2m})\ldots L_\mathrm{I}(t_2)L_\mathrm{I}(t_1)(\rho_{\mathrm{S},0}\otimes P_\Omega)\right]. \tag{2.48}$$

Since the operators L_I are unbounded, the formula and its derivation require justification that we provide in [4]. The integrand can be written in terms of the formalism developed in Section 2.1.1 with obvious modifications

$$\mathrm{Tr}_\mathrm{F}\left[L_\mathrm{I}(t_{2m})\ldots L_\mathrm{I}(t_2)L_\mathrm{I}(t_1)(\rho_{\mathrm{S},0}\otimes P_\Omega)\right] = \left(\mathcal{T}\mathbb{E}\left[L_\mathrm{I}(t_{2m})\otimes_\mathrm{S}\ldots\otimes_\mathrm{S} L_\mathrm{I}(t_2)\otimes_\mathrm{S} L_\mathrm{I}(t_1)\right]\right)\rho_{\mathrm{S},0}. \tag{2.49}$$

These modifications will not be discussed here in detail (see [4]). Briefly said, we introduce copies of $\mathscr{R}$ indexed by the times $t_1,t_2,\ldots,t_m$ and labelled products of them. For example, the term $\mathbb{E}[\ldots]$ above is an element of $\mathscr{R}^{\otimes^m}$ that we identify with an element of $\mathscr{R}_{\{t_1,\ldots,t_m\}}$, and the operator $\mathcal{T}$ contracts it into an element of $\mathscr{R}$. Applying Wick's theorem, one gets

$$(-1)^m\mathbb{E}\left[L_\mathrm{I}(t_{2m})\otimes_\mathrm{S}\ldots\otimes_\mathrm{S} L_\mathrm{I}(t_2)\otimes_\mathrm{S} L_\mathrm{I}(t_1)\right] = \sum_{\pi\in\mathrm{Pair}(t_1,\ldots,t_{2m})}\ \bigotimes_{(u,v)\in\pi} K_{u,v} \tag{2.50}$$

where $K_{u,v}$ is defined as an operator in $\mathscr{R}^{\otimes^2}$:

$$K_{u,v} = -\mathbb{E}(L_\mathrm{I}(v)\otimes_\mathrm{S} L_\mathrm{I}(u)), \tag{2.51}$$

identified with an element of $\mathscr{R}_{\{u,v\}}$, and $\mathrm{Pair}(t_1,\dots,t_{2m})$ denotes the set of pairings of the set $\{t_1,\dots,t_{2m}\}$, and we write the pairs as ordered pairs (u,v) with the convention $u\le v$. Substituting (2.50) in (2.48) we arrive at

$$\mathrm{e}^{\mathrm{i}tL_{\mathrm{S}}}Q_n=\sum_{m\in\mathbb{N}}\int_{-1\le t_1<\ldots<t_{2m}<n/\lambda^2}\mathrm{d}t_1\ldots\mathrm{d}t_{2m}\sum_{\pi\in\mathrm{Pair}(t_1,\ldots,t_{2m})}\mathcal{T}\left[\bigotimes_{w\in\pi}K_w\right] \tag{2.52}$$

where we abbreviate the pairs as $w=(u,v)$. In [4] it is explained how this expression may be written as an integral in a suitable space. Consider a set whose elements are families $\underline{w}$ of pairs of times: $\underline{w}=\{w_1,w_2,\dots,w_m\}$ with $m\ge 0$ and $w_i=(u_i,v_i)$, $u_i\le v_i$ and $u_i,v_i\in[-1,n/\lambda^2]$. This set carries a σ-algebra and a measure $\mu(\mathrm{d}\underline{w})$ so that (2.52) becomes

$$\mathrm{e}^{\mathrm{i}tL_{\mathrm{S}}}Q_n=\int\mu(\mathrm{d}\underline{w})\mathcal{T}\left[\bigotimes_i K_{w_i}\right]. \tag{2.53}$$

It is understood that $\underline{w}=\emptyset$ contributes 1 to the right hand side. Let us now additionally define

$$K_{w|b}:=-\mathbb{E}(U_{n+1}\otimes_{\mathrm{S}}L_{\mathrm{I}}(v)\otimes_{\mathrm{S}}L_{\mathrm{I}}(u)),\qquad w=(u,v), \tag{2.54}$$

as an operator in $\mathscr{R}^{\otimes^3}$ that we identify with $\mathscr{R}_{\{u,v,t\}}$ (recall $t=n/\lambda^2$) such that the operator U_{n+1} acts on the space indexed by t. Then, the expansion (2.52) can also be performed in the presence of the observable $\mathrm{d}\Gamma(b)$:

$$\begin{aligned}\mathrm{e}^{\mathrm{i}tL_{\mathrm{S}}}Q_{n|b}&=\sum_{m\in\mathbb{N}}(-1)^m\int_{-1\le t_1<\ldots<t_{2m}<n/\lambda^2}\mathrm{d}t_1\ldots\mathrm{d}t_{2m}\ \mathrm{Tr}_{\mathrm{F}}\left[U_{n+1}L_{\mathrm{I}}(t_{2m})\ldots L_{\mathrm{I}}(t_1)(\cdot\otimes P_\Omega)\right]\\&=\sum_{m\in\mathbb{N}}\int_{-1\le t_1<\cdots<t_{2m}\le n/\lambda^2}\mathrm{d}t_1\ldots\mathrm{d}t_{2m}\sum_{\pi\in\mathrm{Pair}(t_1,\ldots,t_{2m})}\sum_{w_0\in\pi}\mathcal{T}\left[K_{w_0|b}\bigotimes_{\substack{w\in\pi\\ w\neq w_0}}K_w\right]\\&=\int\mu(\mathrm{d}\underline{w})\sum_i\mathcal{T}\left[K_{w_i|b}\bigotimes_{j\neq i}K_{w_j}\right].\end{aligned} \tag{2.55}$$

We proceed with the identification of G_A^{c} from these expansions. To do that we need to coarse grain them to the macroscopic time scale (in units of $1/\lambda^2$). Given an $s\in[-1,n/\lambda^2]$ let $[s]$ denote the smallest integer not smaller than $\lambda^2 s$ i.e. $s\in]\lambda^{-2}([s]-1),\lambda^{-2}[s]]$. Then, given $\underline{w}=\{w_1,w_2,\dots,w_m\}$ let $[\underline{w}]\subset\mathbb{N}$ be the union of the $[u_i]$ and $[v_i]$ for $w_i=(u_i,v_i)$.

The contraction operator $\mathcal{T}[\cdot]$ defined in Section 2.1.3 contracts operators from $\mathscr{R}_A$ to $\mathscr{R}$. We now define a contraction operator $\mathcal{T}_A$ that produces operators in $\mathscr{R}_A$. Let us consider a finite family of operators $V_{t_i}\in\mathscr{R}_{t_i}$ where the indexed times t_i satisfy $t_i<t_{i+1}$ and $[t_i]\in A$. Then we set

$$\mathcal{T}_A\left[\bigotimes_i V_{t_i}\right]:=\bigotimes_{\tau\in A}\boldsymbol{I}_\tau\left[\mathcal{T}\left[\bigotimes_{j:[t_j]=\tau}V_{t_j}\right]\right] \tag{2.56}$$

and we extend by linearity to the whole of $\otimes_i\mathscr{R}_{t_i}$, obtaining $\mathcal{T}_A:\otimes_i\mathscr{R}_{t_i}\mapsto\mathscr{R}_A$. In words, $\mathcal{T}_A$ puts each operator into the right 'macroscopic' time-copy and contracts the operators within each macroscopic time-copy. Coarsegraining (2.53) this way leads to the formula

$$\tilde{Y}_A G_A Y_A=\int\mu(\mathrm{d}\underline{w})1_{[\underline{w}]=A}\mathcal{T}_A\left[\bigotimes_i K_{w_i}\right]. \tag{2.57}$$

The factors $\tilde{Y}_A$ and Y_A come from the free S-evolutions in (2.53) and the definition of G_A. They are defined as

$$Y_A=\bigotimes_{\tau\in A\setminus\{0\}}Y_\tau,\qquad \widetilde{Y}_A=\bigotimes_{\tau\in A\setminus\{0\}}\widetilde{Y}_\tau, \tag{2.58}$$

with

$$Y_\tau=\boldsymbol{I}_\tau[\mathrm{e}^{\mathrm{i}(\tau-1)L_{\mathrm{S}}}],\qquad \widetilde{Y}_\tau=\boldsymbol{I}_\tau[\mathrm{e}^{-\mathrm{i}\tau L_{\mathrm{S}}}]. \tag{2.59}$$

Since $\mathrm{e}^{-\mathrm{i}\tau L_{\mathrm{S}}}$ is an isometry in the operator norm of $\mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$, left and right multiplication by $Y_A,\widetilde{Y}_A$ is an isometry on $\mathscr{R}_A$ in the norm $\|\cdot\|_\diamond$, and therefore $\widetilde{Y}_A$ and Y_A play no role in what follows.

The connected correlations G^c_A have similar quite obvious expressions. Given a $\underline{w}$ we can define an undirected graph $\mathcal{G}(\underline{w})$ with vertex set $[\underline{w}]$ and edges $\{\tau,\tau'\},\tau\leq\tau'$ whenever there is a pair $w_i=(u_i,v_i)$ such that $[u_i]=\tau$ and $[v_i]=\tau'$. Let moreover

$$\mathcal{C}(A):=\{\underline{w}\,\big|\,[\underline{w}]=A \text{ and } \mathcal{G}(\underline{w}) \text{ is connected}\},\tag{2.60}$$

$$\mathcal{C}'(A):=\begin{cases}\{\underline{w}\,\big|\,[\underline{w}]=A\} & |A|=1\\ \mathcal{C}(A) & |A|>1.\end{cases}\tag{2.61}$$

We have then

Lemma 2.4. *Let $A\in I_{0,n}$. Then*

$$G^c_A\cong\int\mu(\mathrm{d}\underline{w})1_{\mathcal{C}(A)}\mathcal{T}_A\left[\bigotimes_i K_{w_i}\right],\tag{2.62}$$

$$G^c_{A\cup\{n+1\}}\cong\int\mu(\mathrm{d}\underline{w})1_{\mathcal{C}'(A)}\sum_i\mathcal{T}_A\left[K_{w_i|b}\bigotimes_{j\neq i}K_{w_j}\right]\tag{2.63}$$

where $\cong$ denotes an isometry in the norm $\|\cdot\|_\diamond$.

The obvious proof of (2.62) is in [4], the proof of (2.63) is analogous. By Lemma 2.4, and the properties of the norm $\|\cdot\|_\diamond$, we immediately get the bounds

$$\|G^c_A\|_\diamond\leq\int\mu(\mathrm{d}\underline{w})1_{\mathcal{C}(A)}\prod_{i=1}^m\|K_{w_i}\|_\diamond,\tag{2.64}$$

$$\|G^c_{A\cup\{n+1\}}\|_\diamond\leq\int\mu(\mathrm{d}\underline{w})1_{\widetilde{\mathcal{C}}(A)}\sum_{i=1}^m\|K_{w_i|b}\|_\diamond\prod_{j\neq i}\|K_{w_j}\|_\diamond.\tag{2.65}$$

2.3.2 Bounds on the operators $K_w, K_{w|b}$ and G^c_A

To bound the operators K_w, we first have to address the fact that these operators are qualitatively different whenever one or both of the times $\{u,v\}$ is smaller than 0 (because then it originates from the expansion of the Weyl operator, rather than from the interaction). Let us write (recall the form factor ϕ)

$$\phi_s=1_{s\geq 0}\mathrm{e}^{\mathrm{i}s\omega}\phi+1_{s<0}\psi_\ltimes.\tag{2.66}$$

Then we define the functions $h(u,v)$ and $h(u,v|b)$ by

$$|\lambda|^{1_{u\geq0}+1_{v\geq0}}h(u,v):=\|K_{u,v}\|_\diamond,\qquad|\lambda|^{1_{u\geq0}+1_{v\geq0}}h(u,v|b):=\|K_{u,v|b}\|_\diamond\tag{2.67}$$

where we should however keep in mind that $h(u,v|b_k)$ depends on δ and $h(u,v|b_x)$ depends on t_c and the final time t. We usually do not indicate this dependence (see however item 1) of Proposition 2.5). From the definition of the norm $\|\cdot\|_\diamond$ and the definition of U_{n+1} in (2.8) we have

$$h(u,v)\leq4\|D\|^2|\langle\phi_v,\phi_u\rangle_{\mathfrak{h}}|,\qquad h(u,v|b)\leq4\|D\|^2|\langle\phi_v,b(t)\phi_u\rangle_{\mathfrak{h}}|\tag{2.68}$$

where $b(t)=\mathrm{e}^{\mathrm{i}\omega t}b\mathrm{e}^{-\mathrm{i}\omega t}$. The important properties of the functions $h(u,v),h(u,v|b)$ are collected in

Proposition 2.5 (Bounds on correlation functions). *Unless mentioned otherwise, let $u>-1$.*

1. *If $u\geq0$ and $s\geq0$, then*

$$h(u,v)=h(u+s,v+s),\qquad h(u,v|b_k)=h(u+s,v+s|b_k).\tag{2.69}$$

The function $h(u,v|b_x)$ depends on the final time t and in general $h(u,v|b_x)\neq h(u+s,v+s|b_x)$. We can indicate this dependence by writing $h(u,v,t|b_x)$, then

$$h(u,v,t|b_x)=h(u+s,v+s,t+s|b_x).\tag{2.70}$$

2.

$$\int_u^t \mathrm{d}v \langle v-u\rangle^{1+\alpha} h(u,v) \le C 1_{u\ge 0} + \breve{C} 1_{u<0}. \tag{2.71}$$

3.

$$\int_u^t \mathrm{d}v \langle v-u\rangle^{1+\alpha/2} h(u,v|b_k) \le \breve{C}\delta^{\alpha/2}. \tag{2.72}$$

4.

$$\int_u^t \mathrm{d}v \langle v-u\rangle^{1+\alpha} h(u,v|b_x)_{\sup_t} \le \breve{C}. \tag{2.73}$$

where $h(u,v|b_x)_{\sup_t} := \sup_{q\in\mathbb{R}} \left(h(u+q, v+q|b_x) 1_{v+q\le t} 1_{u+q\ge -1}\right)$.

5. Recall $r_\theta < 1$ *is the radius of a ball containing* $\mathrm{Supp}\theta$*. Fix a number* m_θ *such that* $r_\theta < m_\theta < 1$*, then*

$$\int_{-1}^{t-m_\theta t_c} \mathrm{d}u \int_u^t \mathrm{d}v \, \langle t - m_\theta t_c - u\rangle^{\alpha} h(u,v|b_x) \le \breve{C} \tag{2.74}$$

where $\breve{C}$ *can depend on* m_θ*.*

Whenever applicable, the bounds above are uniform in u *and* t*.*

Note that in item 4), $h(u,v|b_x)_{\sup_t}$ differs from $h(u,v|b_x)$ in that; unlike the latter, it is a function of $v-u$, i.e. it is translation invariant. The same remark applies to the upcoming bounds (2.85) and (2.90). Item 1) follows immediately from the fact that $\theta(k/\delta)$ commutes with $\mathrm{e}^{\mathrm{i}s\omega}, s\in\mathbb{R}$ and the group property $\mathrm{e}^{\mathrm{i}s\omega}\mathrm{e}^{\mathrm{i}s'\omega} = \mathrm{e}^{\mathrm{i}(s+s')\omega}$. The proofs of the other claims concern only the one-boson problem and they are of a completely different nature than the rest of this paper. Therefore, we gather those proofs in Appendix A.

2.3.3 Bounds for operator-valued polymers

The next step is to use the bounds on the h-functions and the formulae (2.64,2.65) to derive bounds on the operator-valued correlation functions G^c_A.

Consider first (2.64). Let $\mathscr{T}(A)$ be the set of $\underline{w}$ such that $[\underline{w}] = A$ and $\mathcal{G}(\underline{w})$ is a (connected) tree. Then

$$\int \mu(\mathrm{d}\underline{w}) 1_{\mathcal{C}(A)} \prod_{i=1}^m \|K_{w_i}\|_\diamond \le \int \mu(\mathrm{d}\underline{w}') 1_{\mathscr{T}(A)} \prod_{i=1}^{m'} \|K_{w_i'}\|_\diamond \int \mu(\mathrm{d}\underline{w}'') 1_{[\underline{w}'']\subset A} \prod_{i=1}^{m''} \|K_{w_i''}\|_\diamond. \tag{2.75}$$

Indeed, the pairings and integrals on the left hand side form a subset of the ones on the right hand side: since $\mathcal{G}(\underline{w})$ is connected it contains a (in general not unique) spanning tree $\mathscr{T}$ (a tree with the same vertex set as the total graph, i.e. $\mathcal{G}(\underline{w})$) and thus there is a subset $\underline{w}'$ of $\underline{w}$ so that $\mathcal{G}(\underline{w}') = \mathscr{T}$. The remaining set of pairs $\underline{w}''$ in $\underline{w}$ meets the constraint $[\underline{w}''] \subset A$.

We first perform the integral over $\underline{w}''$. The integrability results in Proposition 2.5 lead to the estimate

$$\int \mu(\mathrm{d}\underline{w}'') 1_{[\underline{w}'']\subset A} \prod_{i=1}^{m''} \|K_{w_i''}\|_\diamond \le (1 + C_{\ltimes} 1_{0\in A}) \mathrm{e}^{C|A|}. \tag{2.76}$$

This is explained in detail in [4] (see the proofs of Lemma 3.1 and Lemma 3.4 therein). To perform the integral over $\underline{w}'$ let us define for $\tau,\tau' \in \mathbb{N}$, $\tau < \tau'$

$$\hat{e}(\tau,\tau') = |\lambda|^{1+1_{\tau>0}} \int_{\mathrm{Dom}(\tau)} \mathrm{d}u \int_{\mathrm{Dom}(\tau')} \mathrm{d}v \, h(u,v). \tag{2.77}$$

Here we use the notation $\mathrm{Dom}(\tau) =]\lambda^{-2}(\tau-1), \lambda^{-2}\tau]$ for $\tau > 0$ and $\mathrm{Dom}(0) =]-1,0]$, i.e. $\mathrm{Dom}(\tau) = \{s \ge -1, [s] = \tau\}$. Then

$$\int \mu(\mathrm{d}\underline{w}') 1_{\mathscr{T}(A)} \prod_{i=1}^{m'} \|K_{w_i'}\|_\diamond \le \sum_{\mathscr{T}:\mathcal{V}(\mathscr{T})=A} \prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})} \hat{e}(\tau,\tau') \tag{2.78}$$

where the sum runs over all trees $\mathscr{T}$ whose vertex set $\mathcal{V}(\mathscr{T})$ is A, i.e. over all spanning trees on A, and $\mathcal{E}(\mathscr{T})$ is the edge set of the tree $\mathscr{T}$. Altogether we have obtained

Lemma 2.6. *Let $A \subset I_{0,n}$, then*

$$\|G^c_A\|_\diamond \le (1 + 1_{0\in A}\breve{C})\mathrm{e}^{C|A|} \sum_{\mathscr{T}:\mathcal{V}(\mathscr{T})=A} \prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})} \hat{e}(\tau,\tau'). \tag{2.79}$$

Let us now derive an analogous bound for $\|G^c_{A\cup\{n+1\}}\|_\diamond$. We first define, for $\tau \le \tau'$ (contrary to the above we will need the case $\tau = \tau'$);

$$\hat{e}(\tau,\tau'|b) := |\lambda|^{1_{\tau>0}+1_{\tau'>0}} \int_{\mathrm{Dom}(\tau)} \mathrm{d}u \int_{\mathrm{Dom}(\tau')} \mathrm{d}v\, h(u,v|b) 1_{v\ge u}. \tag{2.80}$$

In (2.65), we distinguish the cases where $[w_i] = \{\tau,\tau'\}, \tau \ne \tau'$ and $[w_i] = \tau_0$. In the first case, we make the edge $\{\tau,\tau'\}$ part of the spanning tree, in the second case we add the factor $\hat{e}(\tau_0,\tau_0|b)$ by hand to the product of edge factors of the spanning tree. The resulting estimate is

Lemma 2.7. *Let $A \subset I_{0,n}$ with $|A| > 1$, then*

$$\begin{aligned}\|G^c_{A\cup\{n+1\}}\|_\diamond \le \breve{C}\mathrm{e}^{C|A|} \sum_{\mathscr{T}:\mathcal{V}(\mathscr{T})=A} \Big(&\sum_{\{\tau_0,\tau_0'\}\in\mathcal{E}(\mathscr{T})} \hat{e}(\tau_0,\tau_0'|b) \prod_{\substack{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})\\ \{\tau,\tau'\}\ne\{\tau_0,\tau_0'\}}} \hat{e}(\tau,\tau') \\ &+ \sum_{\tau_0\in A} \hat{e}(\tau_0,\tau_0|b) \prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})} \hat{e}(\tau,\tau')\Big).\end{aligned} \tag{2.81}$$

In case $A = \{\tau\}$, we have simply $\|G^c_{\{\tau,n+1\}}\|_\diamond \le \breve{C}\hat{e}(\tau,\tau|b)$.

To proceed, we need bounds on the $\hat{e}$ factors. They follow rather straightforwardly from the bounds on $h(u,v), h(u,v|b)$. For convenience we set $\hat{e}(\tau',\tau) := \hat{e}(\tau,\tau')$ and $\hat{e}(\tau',\tau|b) := \hat{e}(\tau,\tau'|b)$. For $\hat{e}(\cdot,\cdot)$, we repeat the bound from [4].

$$\sum_{\tau'\in I_{1,n}\setminus\{\tau\}} \langle\tau'-\tau\rangle^{1+\alpha}\hat{e}(\tau,\tau') \le \begin{cases} C\lambda^2 & \tau \ne 0 \\ \breve{C}|\lambda| & \tau = 0. \end{cases} \tag{2.82}$$

To obtain this bound, we bound the sum by (a constant times) the integrals $\int \mathrm{d}u \int \mathrm{d}v$. For $|\tau'-\tau| > 1$, we gain a factor $|\lambda|^{2(1+\alpha)}$ by using $\langle\tau'-\tau\rangle^{1+\alpha} \le |\lambda|^{2(1+\alpha)}\langle v-u\rangle^{1+\alpha}$ and item 2) of Proposition 2.5. This factor compensates the λ^{-2} coming from the integration over u (in case $\tau > 0$) so that the explicit $|\lambda|^{1+1_{\tau>0}}$ factor from (2.77) is retained on the right hand side of (2.82). For $\tau' = \tau+1$, we estimate (for definiteness, take $\tau > 0$, the other case is trivial)

$$\int_{(\tau-1)/\lambda^2}^{\tau/\lambda^2} \mathrm{d}u \int_{\tau/\lambda^2}^{(\tau+1)/\lambda^2} \mathrm{d}v\, h(u,v) \le \int_0^\infty \mathrm{d}v |v|\, h(0,v) \le C. \tag{2.83}$$

where we used translation invariance (Item 1) of Proposition 2.5). For $b = b_k$, we get

$$\sum_{\tau'\in I_{0,n}} \langle\tau'-\tau\rangle^{1+\alpha/2}\hat{e}(\tau,\tau'|b_k) \le \breve{C}\delta^{\alpha/2}. \tag{2.84}$$

Compared to (2.82), the term $\tau = \tau'$ is now included in the sum. The derivation proceeds as above, but now starting from items 1,3) of Proposition 2.5. In case $\tau' = \tau$, one cannot extract any λ-dependent small factor from the change of variables $(\tau,\tau') \to (u,v)$ so that the explicit $|\lambda|^{1_{\tau'>0}+1_{\tau>0}}$ factor from (2.80) is used to cancel the u,v integration and therefore there is no λ-dependent small factor on the right hand side of (2.84).

For $b = b_x$, we similarly derive the analogue of (2.84), using Proposition 2.5, item 4);

$$\sum_{\tau'\in I_{0,n}} \langle\tau'-\tau\rangle^{1+\alpha}\hat{e}(\tau,\tau'|b_x)_{\sup_n} \le \breve{C} \tag{2.85}$$

where $\hat{e}(\tau,\tau'|b_x)_{\sup_n} := \sup_{\tau''\in\mathbb{Z}} \big(\hat{e}(\tau+\tau'',\tau'+\tau''|b_x)1_{\tau+\tau''\ge 0}1_{\tau'+\tau''\le n}\big)$. Using Proposition 2.5, item 5), we also get

$$\sum_{\tau\le n-m_\theta n_c} \sum_{\tau\le\tau'\le n} \langle n-m_\theta n_c-\tau\rangle^\alpha \hat{e}(\tau,\tau'|b_x) \le \breve{C} \tag{2.86}$$

where n_c was defined at the beginning of Section 2 (recall $t_c = \lambda^{-2}n_c$).

2.4 Properties of scalar polymers

The scalar polymer weights $v(A)$ were defined in Section 2.2. We state some bounds.

Lemma 2.8 (Bounds on scalar polymers)**.** *All estimates hold uniformly in τ.*

1) *For bulk polymers, i.e. $0 \notin A$, we have for $\tau \in I_{1,n}$,*

$$\sum_{A \subset I_{1,n} \colon \tau \in A} \mathrm{e}^{c|A|} d(A)^{1+\alpha} |v(A)| \leq C\lambda^2. \tag{2.87}$$

2) *For polymers containing 0, we have*

$$\sum_{A \subset I_{0,n} \colon 0 \in A} \mathrm{e}^{c|A|} d(A)^{1+\alpha} |v(A)| \leq \check{C}|\lambda|. \tag{2.88}$$

3) *Let $b = b_k$, then for $\tau \in I_{0,n}$,*

$$\sum_{A \subset I_{n,0} \colon \tau \in A} \mathrm{e}^{c|A|} d(A)^{1+\alpha/2} |v(A \cup \{n+1\})| \leq \check{C}\delta^{\alpha/2}. \tag{2.89}$$

4) *Let $b = b_x$, then for $\tau \in I_{0,n}$,*

$$\sum_{A \subset I_{0,n} \colon \tau \in A} \mathrm{e}^{c|A|} d(A)^{1+\alpha} |v(A \cup \{n+1\})|_{\sup_n} \leq \check{C} \tag{2.90}$$

where $|v(A \cup \{n+1\})|_{\sup_n} := \sup_{\tau' \in \mathbb{Z}} \left(|v((A+\tau') \cup \{n+1\})| \, 1_{\min(A+\tau') \geq 0} 1_{\max(A+\tau') \leq n} \right)$.

5) *Let $b = b_x$, then*

$$\sum_{A \subset I_{0,n} : \min A \leq n - m_\theta n_c} \mathrm{e}^{c|A|} \langle n - m_\theta n_c - \min A \rangle^{\alpha} |v(A \cup \{n+1\})| \leq \check{C}. \tag{2.91}$$

2.4.1 Proof of Lemma 2.8

First, we restrict to $A \subset I_{1,n}$. By using the definitions (2.40, 2.39, 2.37, 2.36), we can bound the polymer weight $v(A)$ by a product of $\| \cdot \|_\diamond$-norms of operators G^c_A, projections R and $(T^\perp)^{|J|}$, i.e.

$$|v(A)| \leq \sum_{(\mathcal{A}, \mathcal{J}) \in \mathfrak{S}^f_n : \mathrm{Supp}(\mathcal{A} \cup \mathcal{J}) = A} \|R\|^{|\mathrm{Supp}\mathcal{A}|} \prod_{A' \in \mathcal{A}} \|G^c_{A'}\|_\diamond \prod_{J \in \mathcal{J}} \|(T^\perp)^{|J|}\|. \tag{2.92}$$

Next, we use $\|R\| \leq C, \|(T^\perp)^m\| \leq C\mathrm{e}^{-mg}$ and the bounds (2.79) on $\|G^c_{A'}\|_\diamond$ to get

$$|v(A)| \leq \sum_{(\mathcal{A}, \mathcal{J}) \in \mathfrak{S}^f_n : \mathrm{Supp}(\mathcal{A} \cup \mathcal{J}) = A} \prod_{J \in \mathcal{J}} (C\mathrm{e}^{-|J|g}) \prod_{A' \in \mathcal{A}} \mathrm{e}^{C|A'|} \sum_{\mathscr{T} : \mathcal{V}(\mathscr{T}) = A'} \prod_{\{\tau, \tau'\} \in \mathcal{E}(\mathscr{T})} \hat{e}(\tau, \tau'). \tag{2.93}$$

Let us now take $A \subset I_{0,n}$, i.e. we allow $0 \in A$, than the bound (2.93) remains valid if we multiply the right hand side by $1 + 1_{0 \in A}\check{C}$. Indeed, the only changes are 1) at most one of the factors $\|T^\perp\|$ is replaced by $\|(T^\perp)_0\| = C\|T^\perp\|$ and 2) the bound (2.79) has the factor $1 + 1_{0 \in A'}\check{C}$ for at most one of the sets A'.

We estimate (2.93) by viewing the sums on the right hand side as a sum over certain connected graphs. Let $\mathscr{S} = \mathcal{J} \sqcup \mathcal{E}(\mathscr{T})$, i.e. we label the element of $\mathscr{S}$ as intervals (J) or edges (E). The elements of $\mathscr{S}$ are denoted by S, S' and collections of them are denoted by $\mathcal{S}$. We write Supp$S$ to denote the subset of $\mathbb{N}$ defined by S, i.e. S without the interval/edge label, and $\mathrm{Supp}\mathcal{S} = \cup_{S \in \mathcal{S}} \mathrm{Supp}S$. We assign to any $S \in \mathscr{S}$ a weight $w^{(\beta)}_{\mathrm{s}}(S)$, with $\beta > 0$, as follows:

$$w^{(\beta)}_{\mathrm{s}}(S) := c(w) \times \begin{cases} \check{c}(w) \langle \tau \rangle^\beta |\lambda|^{-1} \hat{e}(0, \tau) & S \text{ is the edge } E = \{0, \tau\}, \qquad 0 < \tau \\ \langle \tau' - \tau \rangle^\beta |\lambda|^{-2} \hat{e}(\tau, \tau') & S \text{ is the edge } E = \{\tau, \tau'\}, \qquad 0 < \tau < \tau' \\ |J|^\beta \mathrm{e}^{-(g/2)|J|} & S \text{ is the interval } J \end{cases} \tag{2.94}$$

where g is as in Lemma 2.1 and the constants $c(w), \breve{c}(w)$ will be fixed below. We define an adjacency relation $\underset{\mathrm{s}}{\sim}$ on $\mathscr{S}$ by

$$
\begin{aligned}
J \underset{\mathrm{s}}{\sim} E &\Leftrightarrow \mathrm{dist}(J,E)=1,\\
E \underset{\mathrm{s}}{\sim} E' &\Leftrightarrow E\cap E' \neq \emptyset,\\
J \underset{\mathrm{s}}{\sim} J' &\Leftrightarrow J = J'.
\end{aligned} \tag{2.95}
$$

Then, using (2.82) and $g>0$, we can choose $c(w), \breve{c}(w)$ small enough such that, for any $\beta \leq 1+\alpha$

$$
\sum_{S\in\mathscr{S}:\, S\underset{\mathrm{s}}{\sim} S'} w_{\mathrm{s}}^{(\beta)}(S) \leq 1/\mathrm{e}, \tag{2.96}
$$

uniformly for small enough λ. We now claim that, for sufficiently small $c>0$,

$$
\mathrm{e}^{c|A|} d(A)^{1+\alpha} |v(A)| \leq (\lambda^2 C 1_{0\notin A} + |\lambda| \breve{C} 1_{0\in A}) \sum_{\substack{\mathcal{S}\subset\mathscr{S}:\, \mathrm{Supp}\mathcal{S}=A\\ \mathcal{S}\text{ connected}}} \prod_{S\in\mathcal{S}} w_{\mathrm{s}}^{(1+\alpha)}(S) \tag{2.97}
$$

where $\mathcal{S}$ connected means that the graph with vertex set $\mathcal{S}$ and edges $\{S,S'\}$ if $S \underset{\mathrm{s}}{\sim} S'$, is connected. To check (2.97), note that

1. $\langle \tau-\tau'\rangle\langle\tau'-\tau''\rangle \leq \langle\tau-\tau''\rangle$
2. The right hand side of (2.93) contains, through the edge factors $\hat{e}$, at least one factor λ^2 when $0\notin A$ and at least one factor $|\lambda|$ or λ^2 when $0\in A$. This is because any contributing fusion has $\mathcal{A}\neq\emptyset$, see Remark 2.3. Additional factors $\mathrm{e}^{C|A'|}$ are killed by additional powers of λ^2.
3. The notion of connectedness defined by the relation $\underset{\mathrm{s}}{\sim}$ corresponds to the one on the right hand side of (2.93) in the following sense: We start from a fusion $(\mathcal{A},\mathcal{J})$ and we choose for any $A'\in\mathcal{A}$, a spanning tree $\mathscr{T}_{A'}$ on A'. Then, consider the subset of $\mathscr{S}$ that consists of $\cup_{A'\in\mathcal{A}}\mathcal{E}(\mathscr{T}_{A'})$ and of the intervals $J\in\mathcal{J}$. This subset is connected by the adjacency relation $\underset{\mathrm{s}}{\sim}$.
4. There is at most one edge E containing 0 so we can absorb an eventual $\breve{c}(w)$ into the prefactor $\breve{C}$.

To finish the proof, we invoke a combinatorial bound stating that, provided (2.96) holds, we have, for any $S_0\in\mathscr{S}$,

$$
\sum_{\substack{\mathcal{S}\subset\mathscr{S}:\mathcal{S}\underset{\mathrm{s}}{\sim} S_0\\ \mathcal{S}\text{ connected}}} \prod_{S\in\mathcal{S}} w_{\mathrm{s}}^{(\beta)}(S) \leq 1, \qquad \sum_{\substack{\mathcal{S}\subset\mathscr{S}\\ \mathcal{S}\cup\{S_0\}\text{ connected}}} \prod_{S\in\mathcal{S}} w_{\mathrm{s}}^{(\beta)}(S) \leq \mathrm{e} \tag{2.98}
$$

where $\mathcal{S}\underset{\mathrm{s}}{\sim} S_0$ means that $S\underset{\mathrm{s}}{\sim} S_0$ for at least one $S\in\mathcal{S}$. An extended presentation of (a more general version of) this bound is found in Appendix A of [4], it is a standard ingredient of cluster expansions. From (2.98), we get

$$
\sum_{\substack{\mathcal{S}\subset\mathscr{S}:\, \tau\in\mathrm{Supp}\mathcal{S}\\ \mathcal{S}\text{ connected}}} \prod_{S\in\mathcal{S}} w_{\mathrm{s}}^{(1+\alpha)}(S) \leq C. \tag{2.99}
$$

Indeed, it is straightforward to relate the constraint $\tau\in\mathrm{Supp}\mathcal{S}$ to the adjacency structure defined by $\underset{\mathrm{s}}{\sim}$. For example: pick an arbitrary τ' with $|\tau'-\tau|\geq 2$ and let $E_{\tau''}$ be the edge $\{\tau',\tau''\}$. Then, $\tau\in\mathrm{Supp}\mathcal{S}$ implies that $\mathcal{S}\underset{\mathrm{s}}{\sim} E_{\tau''}$ for at least one $\tau''\in\{\tau-1,\tau,\tau+1\}$, and hence (2.99) follows by the first inequality of (2.98). Combining (2.97) and (2.99) yields item 1) and item 2).

Next, we turn to the case where $(n+1)\in A$. In the simplest case, $A=\{\tau,n+1\}$ for some τ, we have

$$
|v(\{\tau,n+1\})| \leq \breve{C}\hat{e}(\tau,\tau|b) \tag{2.100}
$$

and all claimed properties, i.e. items 3,4,5 follow immediately from properties of $\hat{e}(\tau,\tau|b)$. Let us hence assume that $|A\setminus\{n+1\}|>1$ in the remainder of the proof. Proceeding as in (2.97), we derive

$$\mathrm{e}^{c|A|}|v(A\cup\{n+1\})|\leq \check{C}\sum_{\tau_1,\tau_2\in A,\tau_1\leq\tau_2}\hat{e}(\tau_1,\tau_2|b)\sum_{\mathcal{S}\subset\mathscr{S}}1_{\mathrm{Supp}\mathcal{S}'=A}1_{\mathcal{S}'\text{ connected}}\prod_{S\in\mathcal{S}}w_{\mathrm{s}}^{(0)}(S), \tag{2.101}$$

with $\mathcal{S}'=\mathcal{S}\cup\{\{\tau_1,\tau_2\}\}$ in case $\tau_1\neq\tau_2$ and $\mathcal{S}'=\mathcal{S}$ if $\tau_1=\tau_2$. We did not extract $\lambda^2,|\lambda|$-factors from the right hand side, in contrast to (2.97), because this smallness is anyhow missing in (2.100). Note furthermore that (2.101) remains valid when we multiply the left hand side by $d(A)^\beta$, and, on the right hand side, we replace $\hat{e}(\tau_1,\tau_2|b)$ by $\langle\tau_2-\tau_1\rangle^\beta\hat{e}(\tau_1,\tau_2|b)$ and $w_{\mathrm{s}}^{(0)}$ by $w_{\mathrm{s}}^{(\beta)}$, for $\beta\leq 1+\alpha$. To obtain item 4), we use (2.101) with these replacements, choosing $\beta=1+\alpha$, and additionally replacing $|v(\cdot)|\to|v(\cdot)|_{\sup_n}$ and $\hat{e}(\cdot,\cdot|b_x)\to\hat{e}(\cdot,\cdot|b_x)_{\sup_n}$. Using the same strategy as in the proof of items 1,2), relying on (2.98), we sum over the collections $\mathcal{S}$ and over τ_1,τ_2, using the bound (2.85) for the edge factor $\hat{e}(\tau_1,\tau_2|b_x)$.

To get item 3), we choose $\beta=1+\alpha/2$ and we proceed as previously; the only difference is that we can extract an additional small factor $\delta^{\alpha/2}$ from the edge factor $\hat{e}(\tau,\tau'|b_k)$, i.e. we use (2.84).

Finally, we deal with item 5). We abbreviate $\tilde{n}:=n-m_\theta n_c$. Note that, if we restrict the sum in (2.91) to A such that $\max A>\tilde{n}$, then the desired bound follows from item 4), hence it suffices in the remainder of the proof to restrict the sum to $\max A\leq\tilde{n}$. We perform this proof in a more abstract way than necessary, because at a later stage we will need an analogous estimate. We recast the bound (2.101) as

$$\mathrm{e}^{c|A|}|v(A\cup\{n+1\})|\leq\check{C}\sum_{\substack{A_0,A_1\subset I_{0,n}:\,|A_0|=1,2\\ A_0\cup A_1=A}}x(A_0)z_{A_0}(A_1) \tag{2.102}$$

where we introduced the weights

$$x(A_0):=\hat{e}(\tau_1,\tau_2|b_x),\qquad A_0=\{\tau_1,\tau_2\}\text{ (possibly }\tau_1=\tau_2), \tag{2.103}$$

$$z_{A_0}(A_1):=\sum_{\mathcal{S}\subset\mathscr{S}}1_{\mathrm{Supp}\mathcal{S}=A_1}1_{\mathcal{S}'\text{ connected}}\prod_{S\in\mathcal{S}}w_{\mathrm{s}}^{(0)}(S) \tag{2.104}$$

with $\mathcal{S}'$ as in (2.101), and $z_{A_0}(\emptyset):=1$. The x,z-weights satisfy the properties

a) Let $|a|_+=\max(a,0)$ for $a\in\mathbb{R}$,

$$\sum_{A_1\subset I_{0,n}}\langle|\min A_0-\min A_1|_+\rangle^{1+\alpha}z_{A_0}(A_1)\leq\check{C}|A_0|. \tag{2.105}$$

b)

$$\sum_{A_0\subset I_{0,n}:\,|A_0|=1,2,\,\min A_0\leq\tilde{n}}\langle\tilde{n}-\min A_0\rangle^\alpha\mathrm{e}^{c|A_0|}x(A_0)\leq\check{C}. \tag{2.106}$$

Of course, in the case at hand, the right hand side of (2.105) is simply $\check{C}$ by the constraint on $|A_0|$. Property a) follows by the same reasoning as the proofs of items 1)-4), after writing the constrained sum over A_1 as $\sum_{\tau\in A_0}\sum_{A_1:A_1\ni\tau}$, and property b) is just the bound (2.86). The statement of item 5), restricted to $\max A\leq\tilde{n}$, is now

$$\sum_{\substack{A_0,A_1\subset I_{0,n}:\,|A_0|=1,2,\\ \max(A_0\cup A_1)\leq\tilde{n}}}\langle\tilde{n}-\min(A_0\cup A_1)\rangle^\alpha x(A_0)z_{A_0}(A_1)\leq\tilde{C}. \tag{2.107}$$

Note that

$$\langle\tilde{n}-\min(A_0\cup A_1)\rangle\leq C\langle\tilde{n}-\min A_0\rangle\langle|\min A_0-\min A_1|_+\rangle. \tag{2.108}$$

We substitute this in the left hand side of (2.107) and use property a) to bound the sum over A_1 by $C|A_0|\leq\mathrm{e}^{c|A_0|}$. Then, we perform the sum over A_0 by property b). This proves the inequality (2.107). □

3 Proofs of the main theorems

In this Section, we give the final proof of our main results, Theorems 1.2 and 1.1. First, in Section 3.1, we introduce some general tools, applying to both choices of the operator b. Most importantly, we develop a refinement of the representation (2.45). In Section 3.3, we specialise to the case $b = b_x$ and we prove the minimal velocity estimate, i.e. Theorem 1.2. In Section 3.4, we take $b = b_k$ and we obtain the soft boson bound, i.e. Theorem 1.1.

3.1 General Tools

As announced, we do not distinguish for now between the two different choices for b, except in Lemma 3.1. We start from the representation (2.45) and we introduce some notation to simplify it. We will use the adjacency relation $A \sim A' \Leftrightarrow \mathrm{dist}(A, A') \leq 1$ for subsets of $I_{0,n}$, and extended to subsets of $I_{0,n+1}$ by simply ignoring the element $n+1$, i.e.:

$$A \sim A' \Leftrightarrow \mathrm{dist}(A \setminus \{n+1\}, A' \setminus \{n+1\}) \leq 1, \qquad A, A' \neq \{n+1\}, \tag{3.1}$$

(we never need the case where A or A' is the singleton $\{n+1\}$). As previously, we write $\mathcal{A} \sim A'$ if there is at least one $A \in \mathcal{A}$ such that $A \sim A'$, and $\mathcal{A} \nsim A'$ if there is no $A \in \mathcal{A}$ such that $A \sim A'$.

We recast (2.45) by separating each collection $\mathcal{A}$ into its boundary and bulk polymers;

$$Z_n = \sum_{A_{\rtimes}} v(A_{\rtimes}) Z_{n,A_{\rtimes}} + \sum_{A_{\rtimes}, A_{\ltimes} : A_{\rtimes} \nsim A_{\ltimes}} v(A_{\ltimes}) v(A_{\rtimes}) Z_{n, A_{\rtimes} \cup A_{\ltimes}} + \sum_{A_{\ltimes,\rtimes}} v(A_{\ltimes,\rtimes}) Z_{n, A_{\ltimes,\rtimes}} \tag{3.2}$$

where we abbreviated $Z_n = Z_n(\mathrm{d}\Gamma(b), \rho_0)$ and where $A_{\ltimes}, A_{\rtimes}, A_{\ltimes,\rtimes}$ run over nonempty subsets of $I_{0,n+1}$ that, respectively,

- contain 0 but not $n+1$,
- contain $n+1$ and at least one other element, but not 0.
- contain both 0 and $n+1$.

and the factors $Z_{n,A}$ in (3.2) are defined as

$$Z_{n,A'} := \sum_{\substack{\mathcal{A} \in \mathfrak{B}^1_{1,n} \\ \mathcal{A} \nsim A'}} \prod_{A \in \mathcal{A}} v(A) \tag{3.3}$$

where it is understood that $\mathcal{A} = \emptyset$ contributes 1 to the right hand side. Note that $Z_{n,A}$ depends only on bulk polymer weights. Moreover, by (2.42),

$$1 = Z_n(\mathbb{1}, \rho_0) = Z_{n,\emptyset}, \qquad \text{for } \rho_0 = \eta \otimes P_\Omega. \tag{3.4}$$

As explained in [4], the quantity $Z_{n,A'}$ can be viewed as the partition function of a polymer gas with polymer weights $w(A) \equiv v(A) 1_{[A \nsim A']}$. For λ small enough, the bound (2.87) (a 'Kotecky-Preiss' criterion, in the terminology of [4]) allows us to apply the cluster expansion and obtain

$$\log Z_{n,A'} = \sum_{\mathcal{A} \in \mathfrak{B}_{1,n}} v^T(\mathcal{A}) 1_{[\mathcal{A} \nsim A']} \tag{3.5}$$

where the *truncated* weights $v^T(\cdot)$ are defined as

$$v^T(\mathcal{A}) := \sum_{\mathscr{G} \in \mathfrak{G}^c(\mathcal{A})} (-1)^{|\mathscr{E}(\mathscr{G})|} \prod_{\{A_i, A_j\} \in \mathscr{E}(\mathscr{G})} 1_{[A_i \sim A_j]} \prod_{A_i \in \mathcal{A}} v(A_i) \tag{3.6}$$

where $\mathfrak{G}^c(\mathcal{A})$ is the set of connected graphs with vertex set $\mathcal{A}$, and $\mathscr{E}(\mathscr{G})$ is the edge set of the graph $\mathscr{G}$, see Appendix A of [4] for more details. The only property of the weights $v^T(\cdot)$ that we need here is[3]

$$\sum_{\mathcal{A} \in \mathfrak{B}_{1,n} : \mathcal{A} \sim A} d(\mathcal{A})^{1+\alpha} \mathrm{e}^{c|\mathrm{Supp}\mathcal{A}|} |v^T(\mathcal{A})| \leq C|\lambda^2||A|. \tag{3.7}$$

[3]The property stated in Appendix A of [4] misses the factor $\mathrm{e}^{c|\mathrm{Supp}\mathcal{A}|}$ but this can be easily obtained by redefining $v(A) \to \mathrm{e}^{c|A|} v(A)$ and taking $|\lambda|$ smaller.

Comparing to the expansion of $Z_{n,\emptyset}$ and using $\log Z_{n,\emptyset} = 0$ (see (3.4)), we get

$$\log Z_{n,A'} = \log \frac{Z_{n,A'}}{Z_{n,\emptyset}} = - \sum_{\mathcal{A} \in \mathfrak{B}_{1,n}} v^T(\mathcal{A}) 1_{[\mathcal{A} \sim A']}. \tag{3.8}$$

We now decompose

$$Z_{n,A'} = \sum_{A \subset I_{1,n}} p_{A'}(A) \tag{3.9}$$

with $p_{A'}(\emptyset) = 1$ and, for $A \neq \emptyset$,

$$p_{A'}(A) = \sum_{\substack{\mathfrak{A} \subset \mathfrak{B}_{1,n} \\ \mathrm{Supp}\mathfrak{A} = A}} \mathfrak{p}_{A'}(\mathfrak{A}), \qquad \mathfrak{p}_{A'}(\mathfrak{A}) = \prod_{\mathcal{A} \in \mathfrak{A}} (\mathrm{e}^{-v^T(\mathcal{A})} - 1) 1_{\mathcal{A} \sim A'} \tag{3.10}$$

with $\mathrm{Supp}\mathfrak{A} = \cup_{\mathcal{A} \in \mathfrak{A}} \mathrm{Supp}\mathcal{A}$. The decomposition (3.9) follows from the identity

$$\prod_{x \in X} \mathrm{e}^{f(x)} = \sum_{Y \subset X} \prod_{x \in Y} (\mathrm{e}^{f(x)} - 1),$$

for a finite set X and $f : X \to \mathbb{C}$ and with $\prod_{x \in \emptyset} := 1$.

Next, we simplify (3.2) by introducing new weights $\bar{v}(\cdot)$. In what follows, A_1 ranges over subsets of $I_{1,n}$, $A_{\ltimes}, A_{\rtimes}, A_{\ltimes,\rtimes}$ have the same meaning as before in (3.2). First we define

$$\bar{v}^{(1)}(A_\sharp) := \sum_{A'_\sharp, A_1 : A'_\sharp \cup A_1 = A_\sharp} v(A'_\sharp) p_{A'_\sharp}(A_1) \tag{3.11}$$

where $(\sharp)$ stands for either one of the three subscripts $(\ltimes), (\rtimes), (\ltimes, \rtimes)$, the same subscript on the left and right hand side of the equation. Then, we also need

$$\bar{v}^{(2)}(A_{\ltimes}) := 0, \qquad \bar{v}^{(2)}(A_{\rtimes}) := 0, \tag{3.12}$$

$$\bar{v}^{(2)}(A_{\ltimes,\rtimes}) := \sum_{\substack{A_{\ltimes}, A_{\rtimes}, A_1 \\ A_{\ltimes} \nsim A_{\rtimes}, A_{\ltimes} \cup A_{\rtimes} \cup A_1 = A_{\ltimes,\rtimes}}} v(A_{\ltimes}) v(A_{\rtimes}) p_{A_{\ltimes} \cup A_{\rtimes}}(A_1) \tag{3.13}$$

and finally

$$\bar{v}(A_\sharp) := \bar{v}^{(1)}(A_\sharp) + \bar{v}^{(2)}(A_\sharp). \tag{3.14}$$

Moreover, we define again $\bar{v}(\{n+1\}) := v(\{n+1\}) = 0$. Relying on (3.9), we recast (3.2) as

$$Z_n = \sum_{A_{\rtimes}} \bar{v}(A_{\rtimes}) + \sum_{A_{\ltimes}, A_{\rtimes} : A_{\ltimes} \nsim A_{\rtimes}} \bar{v}(A_{\ltimes}) \bar{v}(A_{\rtimes}) + \sum_{A_{\ltimes,\rtimes}} \bar{v}(A_{\ltimes,\rtimes}). \tag{3.15}$$

For example, note that the $\bar{v}^{(2)}(\cdot)$ weights account for contributions to the second term of (3.2) that contribute to the third term in (3.15). In other words, if we expand $Z_{n, A_{\ltimes} \cup A_{\rtimes}}$ in the second term of (3.2) according to (3.9), then the terms with $A \sim A_{\ltimes}, A \sim A_{\rtimes}$ contribute to the $\bar{v}^{(2)}(\cdot)$ weights.

Furthermore, we rewrite (3.15) by first remarking that (for any n)

$$\sum_{A_{\ltimes}} \bar{v}(A_{\ltimes}) = 0. \tag{3.16}$$

Indeed, consider (3.15) for $Z_n(\mathbb{1}, \rho_0) = 1$, then polymers A with $n+1 \in A$ never appear and in that case (3.15) simply reads $1 = 1 + \sum_{A_{\ltimes}} \bar{v}(A_{\ltimes})$. Then, we decompose $\sum_{A_{\ltimes} \nsim A_{\rtimes}} = (\sum_{A_{\ltimes}})(\sum_{A_{\rtimes}}) - \sum_{A_{\ltimes} \sim A_{\rtimes}}$ so that we get our final expression

$$Z_n = \sum_{A_{\rtimes}} \bar{v}(A_{\rtimes}) - \sum_{A_{\ltimes} \sim A_{\rtimes}} \bar{v}(A_{\ltimes}) \bar{v}(A_{\rtimes}) + \sum_{A_{\ltimes,\rtimes}} \bar{v}(A_{\ltimes,\rtimes}). \tag{3.17}$$

The weights $\bar{v}(\cdot)$ have analogous properties to the $v(\cdot)$-weights. The time-translation invariance properties will be stated later, here we deal with the bounds:

Lemma 3.1. *Items* $2,3,4,5$ *of Lemma 2.8 hold with* $v(\cdot)$ *replaced by* $\bar{v}(\cdot)$*, possibly with different constants* $c, C, \check{C}$*.*

We will henceforth refer to items $2,3,4,5$ of Lemma 3.1 (There is no item 1) since we did not define $\bar{v}(A)$ for $A \subset I_{1,n}$).

Proof. The $\bar{v}$-weights are built from the v-weights by 'dressing' - in the sense of (3.11, 3.13) - the v-weights with bulk polymers whose p-weights are small and have strong summability properties, as Lemma 3.2 below shows. This should be compared to the proof of Lemma 2.8 where the edge factors $\hat{e}(\tau_1, \tau_2|b)$ were dressed with collections $\mathcal{S}$, cfr. (2.101). We first abbreviate

$$\mathfrak{p}_{A'}(\mathcal{A}) = (\mathrm{e}^{-v^T(\mathcal{A})} - 1)1_{\mathcal{A}\sim A'}, \qquad \text{such that}\, \mathfrak{p}_{A'}(\mathfrak{A}) = \prod_{\mathcal{A}\in\mathfrak{A}} \mathfrak{p}_{A'}(\mathcal{A}) \tag{3.18}$$

and

$$r(A) = d(A)^{1+\alpha}\mathrm{e}^{c|A|}, \qquad r(\mathcal{A}) = r(\mathrm{Supp}\mathcal{A}). \tag{3.19}$$

Then

Lemma 3.2. *For any* $A' \neq \emptyset$*,*

$$\sum_{\mathfrak{A}\subset\mathfrak{B}_{1,n}} \prod_{\mathcal{A}\in\mathfrak{A}} r(\mathcal{A})|\mathfrak{p}_{A'}(\mathcal{A})| \leq \mathrm{e}^{C\lambda^2|A'|}. \tag{3.20}$$

Proof. We bound the left hand side of (3.20) by

$$1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\mathcal{A}_1,\ldots,\mathcal{A}_k\in\mathfrak{B}_{1,n}} \prod_{j=1}^{k} r(\mathcal{A}_j)|\mathfrak{p}_{A'}(\mathcal{A}_j)| \leq \mathrm{e}^{\sum_{\mathcal{A}\in\mathfrak{B}_{1,n}} r(\mathcal{A})|\mathfrak{p}_{A'}(\mathcal{A})|}.$$

The exponent is bounded by $C\lambda^2|A'|$, by (3.7). □

We give sketches of the proofs of items 3) and 5). The remaining items 2) and 4) are treated similarly to 3) and we omit their proofs. We start with item 3). For the sake of simplicity we drop the $\bar{v}^{(2)}(\cdot)$ contribution as its treatment is similar to the $\bar{v}^{(1)}(\cdot)$ contribution. To get item 3) with $\bar{v}(\cdot)$ replaced by $\bar{v}^{(1)}(\cdot)$, it suffices to show

$$\sum_{\substack{A_0\subset I_{0,n},\mathfrak{A}\in\mathfrak{B}_{1,n}\\ \tau\in A_0\cup\mathrm{Supp}\mathfrak{A}}} \Big(\prod_{\mathcal{A}\in\mathfrak{A}} r(\mathcal{A})|p_{A_0}(\mathcal{A})|\Big)\, r(A_0)|v(A_0\cup\{n+1\})| \leq \check{C}. \tag{3.21}$$

We dominate this as $\sum_{A_0,\mathfrak{A}:\tau\in A_0\cup\mathrm{Supp}\mathfrak{A}} \leq \sum_{A_0:\tau\in A_0} + \sum_{\mathfrak{A}:\tau\in\mathrm{Supp}\mathfrak{A}}$. In the first term, we first estimate the sum over $\mathfrak{A}$ by $\mathrm{e}^{C\lambda^2|A_0|}$ using Lemma 3.2, and then we use item 3) of Lemma 2.8 to perform the sum over A_0 with $\tau \in A_0$. In the second term, we pick arbitrarily a $\mathcal{A} \in \mathfrak{A}$ such that $\tau \in \mathrm{Supp}\mathcal{A}$ (hence in particular $\mathcal{A} \sim A_0$ and $\mathcal{A} \sim \{\tau\}$) and we dominate this term by

$$\sum_{\mathcal{A}\in\mathfrak{B}_{1,n}} |\mathfrak{p}_{\{\tau\}}(\mathcal{A})|r(\mathcal{A}) \sum_{A_0\subset I_{0,n}:A_0\sim\mathcal{A}} r(A_0)|v(A_0\cup\{n+1\})| \sum_{\mathfrak{A}'} \prod_{\mathcal{A}'\in\mathfrak{A}'} r(\mathcal{A}')|\mathfrak{p}_{A_0}(\mathcal{A}')|. \tag{3.22}$$

The sum over $\mathfrak{A}'$ is dominated by $\mathrm{e}^{C\lambda^2|A_0|}$ by Lemma 3.2, the sum over A_0 is dominated by $\check{C}|\mathrm{Supp}\mathcal{A}| \leq \check{C}\mathrm{e}^{c|\mathrm{Supp}\mathcal{A}|}$ by item 3) of Lemma 2.8 upon adjusting c, and the final sum over $\mathcal{A}$ is dominated by $\check{C}$ by (3.7), again adjusting c. Finally, we treat item 5). As argued in the proof of item 5) in Lemma 2.8, we can assume $\max A \leq \tilde{n}$ (the case $\max A > \tilde{n}$ being handled by item 4)), hence it suffices to show

$$\sum_{\substack{A_0\in I_{0,n},A_1\subset I_{1,n}\\ \max(A_0\cup A_1)\leq\tilde{n}}} \langle\tilde{n} - \min(A_0\cup A_1)\rangle^{\alpha} x(A_0) z_{A_0}(A_1) \leq \tilde{C} \tag{3.23}$$

with $\tilde{n} = n - n_c m_\theta$ and

$$x(A_0) := \mathrm{e}^{c|A_0|}|v(A_0\cup\{n+1\})|, \qquad z_{A_0}(A_1) := \mathrm{e}^{c|A_1|}|p_{A_0}(A_1)|.$$

Note the similarity of (3.23) with (2.107), the only difference being that here we do not restrict $|A_0|$ and that we have $A_1 \subset I_{1,n}$, i.e. $0 \notin A_1$. With these small changes, the properties $a), b)$ (2.105, 2.106) hold with the x, z weights as defined here: Property a) by (3.10) and Lemma 3.2, and property b) by item 5) of Lemma 2.8. Therefore, we can repeat the short proof given in the proof of item 5) of Lemma 2.8 to get the desired claim. □

3.2 Symmetry properties of the $\bar{v}(\cdot)$ weights

We list some symmetry properties of the $\bar{v}(\cdot)$ weights. We indicate the dependence on the final time explicitly by writing $\bar{v}_n(A)$ instead of $\bar{v}(A)$.

Let first $b = b_k$. For $\tau \in \mathbb{N}$, $n' > n$ and A such that both $A, A+\tau$ are subsets of $I_{1,n}$

$$\bar{v}_n((A+\tau) \cup \{n+1\}) = \bar{v}_n(A \cup \{n+1\}) = \bar{v}_{n'}(A \cup \{n'+1\}). \tag{3.24}$$

Let now $b = b_x$, then these equalities do not hold in general, but we still have

$$\bar{v}_n(A \cup \{n+1\}) = \bar{v}_{n+\tau}((A+\tau) \cup \{n+\tau+1\}) \tag{3.25}$$

where it is understood that n_c is kept fixed. To establish these properties, one first checks that the same properties hold for the $v(\cdot)$ weights. This follows easily from the symmetry properties in Section 2.1.2. Since (unlike the $\bar{v}(\cdot)$ weights) the $v(\cdot)$ weights are also defined for $A \subset I_{1,n}$, we can also state an additional symmetry property, namely, for $\tau \in \mathbb{N}$, $n' > n$ and A such that both $A, A+\tau$ are subsets of $I_{1,n}$,

$$v_n(A+\tau) = v_n(A) = v_{n'}(A). \tag{3.26}$$

Finally the symmetry properties of the $\bar{v}(\cdot)$ weights follow from the corresponding properties for the $v(\cdot)$ weights and from (3.26).

3.3 Minimal velocity estimate

In this section, we take throughout $b = b_x = \theta(x/t_c)$ and we again abbreviate $Z_n = Z_n(\mathrm{d}\Gamma(b_x), \rho_0)$. Assume the same conventions for the sets $A_\ltimes, A_\rtimes$ as above, then we have

Lemma 3.3.

$$\left| Z_n - \sum_{A_\rtimes} \bar{v}(A_\rtimes) \right| \leq \breve{C} \langle n \rangle^{-\alpha}. \tag{3.27}$$

Proof. From (3.17), the expression between $|\cdot|$ equals

$$-\sum_{A_\ltimes \sim A_\rtimes} \bar{v}(A_\ltimes) \bar{v}(A_\rtimes) + \sum_{A_{\ltimes,\rtimes}} \bar{v}(A_{\ltimes,\rtimes}). \tag{3.28}$$

Let us treat the first term. We distinguish the cases $\min A_\rtimes \leq (n - m_\theta n_c)/2$ and $\min A_\rtimes > (n - m_\theta n_c)/2$. In the first case, we first sum over $A_\ltimes$ using item 2) of Lemma 3.1, yielding a factor $\breve{C}$ (in fact $\breve{C}|\lambda|$) and then over $A_\rtimes$, using item 5) of Lemma 3.1 and the fact that

$$n - n_c m_\theta - (n - m_\theta n_c)/2 \geq (1 - m_\theta) n/2 = \breve{c} n,$$

and obtaining $\breve{C} \langle n \rangle^{-\alpha}$ (recall that $n_c \leq n$). In the second case, we first sum over $A_\ltimes$ (item 2) of Lemma 3.1) obtaining a factor $\breve{C} \langle \min A_\rtimes \rangle^{-(1+\alpha)}$ because $\max A_\ltimes \geq \min A_\rtimes - 1$ (since $A_\ltimes \sim A_\rtimes$), then we sum over $A_\rtimes$ keeping $\min A_\rtimes$ fixed, yielding $\breve{C}$ by item 4) of Lemma 3.1, and finally over $\min A_\rtimes$, yielding $\breve{C} \langle (n - m_\theta n_c)/2 \rangle^{-\alpha} = \breve{C} \langle n \rangle^{-\alpha}$. The second term in (3.28) is estimated in an analogous way. $\square$

Now we are ready to consider the limit $n \to \infty$ in the expression for $Z_n(\mathrm{d}\Gamma(b_x), \rho_0)$. Therefore, we should render the n-dependence in the weights explicit, as we did in Section 3.2. Therefore, we introduce new notation, namely, for a finite $A \subset \mathbb{N}_0$,

$$\nu(A) := \bar{v}_n((n+1-A) \cup \{n+1\}), \qquad \text{with } n \text{ such that } A \subset I_{1,n}. \tag{3.29}$$

With this notation, we have (recall that the sum over $A_\rtimes$ runs over sets not including 0, and hence $\bar{v}(A_\rtimes)$ does not depend on ρ_0)

$$\sum_{A_\rtimes} \bar{v}(A_\rtimes) = \sum_{A \subset I_{1,n}} \nu(A). \tag{3.30}$$

Let us define

$$Z_\infty := \sum_{A \subset \mathbb{N} \colon |A| < \infty} \nu(A). \tag{3.31}$$

Note that Z_∞ still depends on t_c (or n_c), but not on ρ_0.

Lemma 3.4. *This sum on the right hand side of* (3.31) *is absolutely convergent and*

$$|Z_n - Z_\infty| \quad \leq \quad \check{C}\langle n\rangle^{-\alpha}. \tag{3.32}$$

Proof. We have

$$\big|Z_\infty - \sum_{A\subset I_{1,n}} \nu(A)\big| \leq \limsup_{n'\to\infty} \sum_{A\subset I_{1,n'}:\,\min A<(n'-n)+1} |\bar{v}_{n'}(A\cup\{n'+1\})| \tag{3.33}$$

$$\leq \check{C}\langle n\rangle^{-\alpha}. \tag{3.34}$$

The first inequality is by (3.29), the second is by item 5) of Lemma 3.1 since $Cn \leq (n' - m_\theta n_c - \min A)$ for any A contributing to the sum. The Lemma then follows by the triangle inequality from (3.34,3.27) and (3.30). ☐

3.3.1 Proof of Theorem 1.2

First, we slightly generalise the setup; we consider $Z_n(O,\rho_0)$ with $O = \mathrm{d}\Gamma(b_x), \mathbb{1}$ and ρ_0 now not longer a density matrix but the rank-1 operator

$$\rho_0 = \big|\psi_{\mathrm{S}} \otimes \mathcal{W}(\psi_\ltimes)\Omega\big\rangle\big\langle\psi'_{\mathrm{S}} \otimes \mathcal{W}(\psi'_\ltimes)\Omega\big| \tag{3.35}$$

with $\psi_{\mathrm{S}}, \psi'_{\mathrm{S}} \in \mathscr{H}_{\mathrm{S}}$ and $\psi_\ltimes, \psi'_\ltimes \in \mathfrak{h}_\alpha$. We can easily go through all arguments, with obvious changes, and get the following analogue of Lemma 3.4

$$|Z_n(\mathrm{d}\Gamma(b_x),\rho_0) - Z_\infty \operatorname{Tr}\rho_0| \leq \check{C}\langle n\rangle^{-\alpha}, \qquad Z_n(\mathbb{1},\rho_0) = \operatorname{Tr}\rho_0 \tag{3.36}$$

with Z_∞ as above in (3.31). Now, take $\Psi \in \mathcal{D}_\alpha$, i.e. a finite linear combination $\Psi = \sum_i \Psi_i$ with $\Psi_i = \psi_{\mathrm{S},i} \otimes \mathcal{W}(\psi_{\ltimes,i})\Omega$, then

$$\lim_{n\to\infty} \langle \Psi(n/\lambda^2), \mathrm{d}\Gamma(b_x)\Psi(n/\lambda^2)\rangle = Z_\infty \sum_{i,j} \langle \Psi_i, \Psi_j\rangle = Z_\infty \|\Psi\|^2. \tag{3.37}$$

Hence, we get the statement of Theorem 1.2 for times t taken along a subsequence n/λ^2 and t_c of the form n_c/λ^2. To get the full statement, we should again generalise the reasoning in a straightforward way.

Assume that the time-discretisation of the model was chosen based on 'mesoscopic time-blocks' of length $\ell|\lambda|^{-2}, \ell \in [1,2]$, instead of $\ell = 1$ as we did previously: this means that we change the definition of $Q_n, Q_{n|b}$ and $U_\tau, \tau = 1,\dots,n$ by replacing $|\lambda|^{-2}$ by $\ell|\lambda|^{-2}$, for example, instead of (2.6), we have

$$U_\tau := \mathrm{e}^{\mathrm{i}\tau(\ell/\lambda^2)L_{\mathrm{F}}}\mathrm{e}^{-\mathrm{i}(\ell/\lambda^2)L}\mathrm{e}^{-\mathrm{i}(\tau-1)(\ell/\lambda^2)L_{\mathrm{F}}}, \qquad \tau \in I_{1,n}. \tag{3.38}$$

Then, Lemma 2.1 holds as well with a constant $C^{(\ell)}$ and gap $g^{(\ell)}$ that can be chosen uniform in $\ell \in [1,2]$, as we easily get from the results in [4], in particular from the proof of Lemma 2.3 1) therein. The rest of the reasoning goes through without any change except for the readjusting of constants. Hence we have now proven Theorem 1.2 restricted to times t taken along a subsequence $n\ell/\lambda^2$ and t_c of the form $n_c\ell/\lambda^2$, and with constants $\check{C}$ on the right hand side that can be chosen uniform in $\ell \in [1,2]$. Finally, t_c can be tuned independently of t by changing the function $\theta(\cdot)$ to $\theta(\ell\cdot)$ for $\ell \in [1,2]$ and again the constants $\check{C}$ can be chosen uniform. This allows to choose any $t_c \geq \lambda^{-2}$ (smaller t_c would require to take ℓ dependent on λ which we prefer to avoid) and to establish the full Theorem 1.2.

3.4 Soft boson bound

In this section, we take to $b = b_k = \theta(k/\delta)$. Recall the conventions for $A_\ltimes, A_\rtimes, A_{\ltimes,\rtimes}$ and the expression (3.17):

$$Z_n = \sum_{A_\rtimes} \bar{v}(A_\rtimes) - \sum_{A_\ltimes \sim A_\rtimes} \bar{v}(A_\ltimes)\bar{v}(A_\rtimes) + \sum_{A_{\ltimes,\rtimes}} \bar{v}(A_{\ltimes,\rtimes}). \tag{3.39}$$

The second and third term on the right hand side are bounded by $\check{C}\delta^{\alpha/2}$, using items 2) and 3) of Lemma 3.1. For the first term on the right hand side, we argue

Lemma 3.5. *There is an n-independent number a such that*

$$\big|\sum_{A_\rtimes} \bar{v}(A_\rtimes) - na\big| \leq \check{C}\delta^{\alpha/2}. \tag{3.40}$$

Proof. To deal with the n -(in)dependence of the weights, we again introduce new notation:

$$\bar{v}(A|b_k) := \bar{v}_n(A \cup \{n+1\}), \qquad \text{where } n \geq \max A, \tag{3.41}$$

and, by (3.24) we have $\bar{v}(A|b_k) = \bar{v}(A+\tau|b_k)$ provided both $A, A+\tau$ are finite subsets of $\mathbb{N}_0$. Then, let us define

$$a := \sum_{A \subset \mathbb{N}_0 \colon |A|<\infty, \min A = 1} \bar{v}(A|b_k) \tag{3.42}$$

where the sum is absolutely convergent by item 3) of Lemma 3.1, and we have

$$na - \sum_{A \subset I_{1,n}} \bar{v}(A|b_k) = \sum_{A \subset \mathbb{N}, \min A = 1} \min(\max A - 1, n) \bar{v}(A|b_k). \tag{3.43}$$

The left hand side is the expression between $|\cdot|$ in (3.40), and the right hand side can be bounded by $\breve{C}\delta^{\alpha/2}$ by using again item 3) of Lemma 3.1. □

By the boson number bound in [4], we know that $\sup_t \langle \Psi_t, N \Psi_t \rangle \leq \breve{C}$, and therefore $\sup_n Z_n(\mathrm{d}\Gamma(b_k), \rho_0) \leq \breve{C}$, see the remark following Theorem 1.1. However, Lemma 3.5 and the bounds on the other terms (second and third) of (3.39) imply that $Z_n(\mathrm{d}\Gamma(b_k), \rho_0) - an$ is uniformly bounded in n. Combining these two statements, we conclude $a = 0$, and therefore, we have shown

$$|Z_n(\mathrm{d}\Gamma(b_k), \rho_0)| \leq \breve{C}\delta^{\alpha/2}. \tag{3.44}$$

We have hence obtained Theorem 1.1 for t restricted to particular vectors $\Psi_0 = \psi_{\mathrm{S}} \otimes \mathcal{W}(\psi_{\ltimes})\Omega$ and times of the form $t = n/\lambda^2$. By the same trick as applied at the end of the proof of Theorem 1.2 in Section 3.3.1 (involving the change of mesoscopic scale $|\lambda|^{-2} \to \ell|\lambda|^{-2}$), we get the statement for all times t. By the Cauchy-Schwarz inequality, we get the statement for any $\Psi \in \mathcal{D}_\alpha$. This proves the full Theorem 1.1.

A One-particle estimates

In this appendix we prove some estimates concerning the dynamics of a single free boson. Very similar estimates were also established by different methods in [7] (the approach here is less elegant, but more self-contained)

Since this section stands apart from the rest of the paper, we we do not rely on previous definitions and conventions, unless explicitly mentioned. In particular, we do not adhere to our earlier convention to distinguish constants C and $\breve{C}$. First, we state

Lemma A.1. *For $f \in L^2(\mathbb{R}^d)$, assume that $\hat{f} \in C^1(\mathbb{R}^d \setminus \{0\})$ such that for some $0 < \gamma < 1$,*

$$|k|^{1-\gamma} \partial \hat{f} \in L^1(\mathbb{R}^d; \mathbb{C}^d), \qquad |k|^{-\gamma} \hat{f} \in L^1(\mathbb{R}^d). \tag{A-1}$$

Then,

$$|f(x)| \leq C(\gamma) |x|^{-\gamma} \left(\| |k|^{-\gamma} \hat{f} \|_1 + \| |k|^{1-\gamma} \partial \hat{f} \|_1 \right).$$

Proof. We write

$$f(x) = \frac{1}{\mathrm{e}^{-\mathrm{i}} - 1} \int \mathrm{d}k \, (\hat{f}(k + \hat{x}/|x|) - \hat{f}(k)) \mathrm{e}^{\mathrm{i}kx}.$$

Divide the integral to $|k| \leq 2|x|^{-1}$ and $|k| > 2|x|^{-1}$. For the first one insert $1 \leq 2^\gamma |x|^{-\gamma} |k|^{-\gamma}$ and the integral is bounded by $2^{1+\gamma} |x|^{-\gamma} \| |k|^{-\gamma} \hat{f} \|_1$. For the second integral, we can assume that $\hat{f}$ is C^1, hence we insert

$$\hat{f}(k + \hat{x}/|x|) - \hat{f}(k) = |x|^{-1} \int_0^1 \mathrm{d}s \, \hat{x} \cdot \partial \hat{f}(k + s\hat{x}/|x|).$$

to bound it by

$$\begin{aligned} \int_0^1 \mathrm{d}s \int_{|k| > \frac{2}{|x|}} \mathrm{d}k \, |x|^{-1} |\partial \hat{f}(k + s\hat{x}/|x|)| \quad &\leq \quad |x|^{-\gamma} \int_{|k| > \frac{1}{|x|}} \mathrm{d}k \, |x|^{-1+\gamma} |\partial \hat{f}(k)| \\ &\leq \quad C |x|^{-\gamma} \| |k|^{1-\gamma} \partial \hat{f} \|_1. \end{aligned} \tag{A-2}$$

since $|x|^{-1+\gamma} \leq |k|^{1-\gamma}$ in the last integral. □

A.1 Minimal velocity estimates

Recall the function θ introduced above Theorem 1.1. It is a spherically symmetric C^∞ function $\mathbb{R}^d \to [0,1]$ with support contained in a ball with radius $r_\theta < 1$ and we write $\theta_s(x) := \theta(x/s)$. Recall also the dense subspace $\mathfrak{h}_\alpha \subset L^2(\mathbb{R}^d)$ and write $\psi_s = \mathrm{e}^{-\mathrm{i}\omega s}\psi$. We prove

Lemma A.2. *Let $d \geq 3$ and $\psi, \psi' \in \mathfrak{h}_\alpha$. There is a $\gamma > \alpha$ such that, for any $s > 0$ and $1 > m_\theta > r_\theta$,*

$$|(\psi'_{s_2}, \theta_s(x)\psi_{s_1})| \leq C\langle s_2 - s_1\rangle^{-2-\gamma}, \tag{A-3}$$

$$\int_{sm_\theta}^\infty \mathrm{d}s_2 \int_0^{s_2} \mathrm{d}s_1\, \langle s_2 - m_\theta s\rangle^\alpha |(\psi'_{s_2}, \theta_s(x)\psi_{s_1})| \leq C \tag{A-4}$$

where C depends on γ, θ and m_θ (in particular it diverges when $m_\theta \to r_\theta$), but not on s.

Upon renaming the time variables, Lemma A.2 yields the claims of Proposition 2.5 with $b = b_x$: Item 4) follows from the bound (A-3) by choosing ψ, ψ' either ϕ or $\psi_\ltimes$, $s = t_c$ and noting that, for example, $\langle \mathrm{e}^{\mathrm{i}v\omega}\phi, b_x(t)\mathrm{e}^{\mathrm{i}u\omega}\phi\rangle$ is the complex conjugate of $\langle \mathrm{e}^{-\mathrm{i}(t-u)\omega}\phi, \theta(x/t_c)\,\mathrm{e}^{-\mathrm{i}(t-v)\omega}\phi\rangle$. Item 5) follows in the same spirit from (A-4). Items 2,3) in Proposition 2.5 are addressed in Section A.2.

A.1.1 Proof of the bound (A-3) in Lemma A.2

We write

$$(\psi'_{s_2}, \theta_s\psi_{s_1}) = \int \mathrm{e}^{\mathrm{i}(|k_2|s_2 - |k_1|s_1)}\hat{\theta}_s(k_1 - k_2)\overline{\hat{\psi'}(k_2)}\hat{\psi}(k_1)\mathrm{d}k_1\mathrm{d}k_2 \tag{A-5}$$

$$= \int_0^\infty \mathrm{d}\omega_1 \int_0^\infty \mathrm{d}\omega_2 \mathrm{e}^{-\mathrm{i}(\omega_+ s_- + \omega_- s_+)} K(\omega_1, \omega_2) \tag{A-6}$$

where $\omega_\pm := \omega_1 \pm \omega_2$, $s_\pm = (s_1 \pm s_2)/2$ and

$$K(\omega_1,\omega_2) = (\omega_1\omega_2)^{d-1}\int_{S^{d-1}} \mathrm{d}\hat{k}_1 \int_{S^{d-1}} \mathrm{d}\hat{k}_2\, \overline{\hat{\psi'}(\omega_2\hat{k}_2)}\hat{\psi}(\omega_1\hat{k}_1)s^d\zeta(s^2(\omega_1\hat{k}_1 - \omega_2\hat{k}_2)^2)$$

where by rotation invariance of θ we have written it as $\hat{\theta}(k) = \zeta(k^2)$ where ζ satisfies

$$|\partial^n\zeta(x)| \leq C(n,N)\langle x\rangle^{-N} \tag{A-7}$$

for all $n, N > 0$.

Lemma A.3. *There is a $\beta > \alpha$ such that for $n = 0,1,2,3$,*

$$|\partial^n_{\omega_+}K(\omega_1,\omega_2)| \leq C(\omega_1^{-n} + \omega_2^{-n})\frac{(\omega_1\omega_2)^{\frac{d+\beta}{2}}}{\langle\omega_1\rangle^2\langle\omega_2\rangle^2}\left(\frac{s(1/s + \omega_+)^{1-d}}{\langle\frac{1}{2}s^2\omega_-^2\rangle^{N/2}} + \frac{s(1/s + |\omega_-|)^{1-d}}{\langle\frac{1}{2}s^2\omega_+^2\rangle^{N/2}}\right). \tag{A-8}$$

Proof. Let $\hat{k}_1 \cdot \hat{k}_2 =: \cos\vartheta$ with $\vartheta \in [0,\pi]$. Then,

$$\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2\cos\vartheta = \cos^2(\tfrac{\vartheta}{2})\omega_-^2 + \sin^2(\tfrac{\vartheta}{2})\omega_+^2 =: Z(\omega_+,\omega_-,\vartheta)$$

and hence

$$K(\omega_1,\omega_2) = s^d\int_0^\pi \mathrm{d}\vartheta\ (\sin\vartheta)^{d-2}\zeta(s^2Z(\omega_+,\omega_-,\vartheta))G(\omega_1,\omega_2,\vartheta)$$

with

$$G(\omega_1,\omega_2,\vartheta) = (\omega_1\omega_2)^{d-1}\int_{S^{d-1}}\mathrm{d}\hat{k}_1\int_{S^{d-2}}\mathrm{d}\hat{p}\,\overline{\hat{\psi'}(\omega_2\hat{k}_2)}\hat{\psi}(\omega_1\hat{k}_1)$$

where $\hat{k}_2 = \sin\vartheta\hat{p} + \cos\vartheta\hat{k}_1$ (and $\hat{p} \perp \hat{k}_1$). Since $\psi, \psi' \in \mathfrak{h}_\alpha$,

$$|\partial^{n_1}_{\omega_1}\partial^{n_2}_{\omega_2}G(\omega_1,\omega_2,\vartheta)| \leq C\prod_{i=1}^2 \omega_i^{\frac{d+\beta}{2} - n_i}\langle\omega_i\rangle^{-2}, \tag{A-9}$$

uniformly in ϑ, for some $\beta > \alpha$. This implies

$$|\partial^n_{\omega_+} G(\omega_1, \omega_2, \vartheta)| \le C(\omega_1^{-n} + \omega_2^{-n}) \prod_{i=1}^{2} \omega_i^{\frac{d+\beta}{2}} \langle \omega_i \rangle^{-2}.$$

From (A-7) we deduce (we abbreviate $Z = Z(\omega_+, \omega_-, \vartheta)$)

$$|\partial^n_{\omega_+} \zeta(s^2 Z)| \le C\omega_+^{-n} \langle s^2 Z\rangle^{-N}.$$

Combining the two previous inequalities, we get

$$|\partial^n_{\omega_+} K(\omega_1, \omega_2)| \le C(\omega_1^{-n} + \omega_2^{-n}) \frac{(\omega_1\omega_2)^{\frac{d+\beta}{2}} H(\omega_1, \omega_2)}{\langle \omega_1\rangle^2 \langle \omega_2 \rangle^2} \tag{A-10}$$

with

$$H(\omega_1, \omega_2) = s^d \int_0^\pi \mathrm{d}\vartheta\ (\sin\vartheta)^{d-2} \langle s^2 Z\rangle^{-N}.$$

For $\vartheta \in [0, \frac{\pi}{2}]$ we have

$$Z \ge \tfrac{1}{2}(\omega_1 - \omega_2)^2 + \tfrac{1}{4}\vartheta^2(\omega_1 + \omega_2)^2$$

and so

$$\langle s^2 Z\rangle^{-N} \le \langle \tfrac{1}{2} s^2 \omega_-^2 \rangle^{-N/2} \langle \tfrac{1}{4} s^2 \vartheta^2 \omega_+^2\rangle^{-N/2}$$

and for $\vartheta \in [\frac{\pi}{2}, \pi]$

$$\langle s^2 Z\rangle^{-N} \le \langle \tfrac{1}{2} s^2 \omega_+^2 \rangle^{-N/2} \langle \tfrac{1}{4} s^2 (\pi-\vartheta)^2 \omega_-^2\rangle^{-N/2}.$$

Since

$$\int_0^{\pi/2} \mathrm{d}\vartheta\ (\sin\vartheta)^{d-2} \langle \tfrac{1}{4} s^2 \vartheta^2 \omega_+^2\rangle^{-N/2} \le C(1 + s\omega_+)^{1-d}$$

and similarly for the integral over $[\pi/2, \pi]$ we get

$$H(\omega_1, \omega_2) \le \frac{s(1/s + \omega_+)^{1-d}}{\langle \frac{1}{2} s^2 \omega_-^2 \rangle^{N/2}} + \frac{s(1/s + \omega_-)^{1-d}}{\langle \frac{1}{2} s^2 \omega_+^2 \rangle^{N/2}}$$

which yields the claim upon substitution in (A-10).

□

Lemma A.3 implies that the functions $\omega_+^{1-\beta} \partial^2_{\omega_+} K(\omega_1, \omega_2)$ and $\omega_+^{\beta} \partial^3_{\omega_+} K(\omega_1, \omega_2)$ are integrable and $\partial_{\omega_+} K(\omega_1, \omega_2)$ vanishes if $\omega_1 = 0$ or $\omega_2 = 0$. Hence

$$(s_2 - s_1)^2 (\psi'_{s_2}, \theta_s \psi_{s_1}) = -4 \int_0^\infty \mathrm{d}\omega_1 \int_0^\infty \mathrm{d}\omega_2 \mathrm{e}^{-\mathrm{i}(\omega_+ s_- + \omega_- s_+)} \partial^2_{\omega_+} K(\omega_1, \omega_2).$$

By Lemma A.1 with $d = 1$, we get, with $0 < \gamma < 1$,

$$|(\psi'_{s_2}, \theta_s \psi_{s_1})| \le C(s_2 - s_1)^{-2-\gamma} \int_0^\infty \mathrm{d}\omega_1 \int_0^\infty \mathrm{d}\omega_2\ \left(|\omega_+^{-\gamma} \partial^2_{\omega_+} K(\omega_1, \omega_2)| + |\omega_+^{1-\gamma} \partial^3_{\omega_+} K(\omega_1, \omega_2)| \right)$$

provided that the right hand side is finite, which we prove now. The contribution of the first term in the second parenthesis in (A-8) is dominated by

$$\int_0^\infty \mathrm{d}\omega_1 \int_0^\infty \mathrm{d}\omega_2 \frac{\omega_1^{\frac{d+\beta}{2}-2-\gamma} \omega_2^{\frac{d+\beta}{2}}}{\langle \omega_1 \rangle^2 \langle \omega_2\rangle^2} \frac{s(1/s + \omega_+)^{1-d}}{\langle \frac{1}{2} s^2 \omega_-^2\rangle^{N/2}} \quad + \quad (\omega_1 \leftrightarrow \omega_2) \tag{A-11}$$

where $(\omega_1 \leftrightarrow \omega_2)$ stands for the same term but with ω_1, ω_2 interchanged. Since this term is treated in the same way, we drop it. Then, the ω_2 integral in (A-11) gives the bound

$$\int_0^\infty \mathrm{d}\omega_2 \frac{\omega_2^{\frac{d+\beta}{2}}}{\langle \omega_2 \rangle^2} \frac{s(1/s + \omega_+)^{1-d}}{\langle \frac{1}{2} s^2 \omega_-^2\rangle^{N/2}} \le C(\omega_1 + 1/s)^{\frac{2+\beta-d}{2}}.$$

Indeed, for large N the $\langle \frac{1}{2}s^2\omega_-^2\rangle^{-N/2}$ factor fixes $\omega_2 = \omega_1 + \mathcal{O}(1/s)$. Therefore, we have

$$(\text{A-11}) \leq C\int_0^\infty \mathrm{d}\omega_1 \frac{\omega_1^{\beta-\gamma-1}}{\langle\omega_1\rangle^2}\Big(\frac{\omega_1}{\omega_1+1/s}\Big)^{\frac{d-2-\beta}{2}} \leq C$$

(uniformly in s) if $\gamma < \beta$ because $d \geq 3$.

The second term between brackets in (A-8) is bounded uniformly in s for $\gamma < \beta$, so the bound (A-3) is proven.

A.1.2 Proof of the bound (A-4) in Lemma A.2

For $\psi \in \mathfrak{h}_\alpha$, we write

$$\psi_s(x) = \int_{S^{d-1}} \mathrm{d}\hat{k}\int_0^\infty \mathrm{d}\omega\, \omega^{d-1}\mathrm{e}^{-\mathrm{i}\omega(s+\hat{k}\cdot x)}\hat{\psi}(\omega\hat{k}),$$

and, by Lemma A.1, for some $\beta > \alpha$,

$$|\psi_s(x)| \leq C\int_{S^{d-1}} \mathrm{d}\hat{k}\ \langle s+\hat{k}\cdot x\rangle^{-d+\frac{1-\beta}{2}}.$$

Therefore, for $\psi, \psi' \in \mathfrak{h}_\alpha$ and m'_θ such that $r_\theta < m'_\theta < m_\theta$

$$|(\psi_{s_1}, \theta_s(x)\psi'_{s_2})| \leq Cs^d\langle s_1 - m'_\theta s\rangle^{-d+\frac{1-\beta}{2}}\langle s_2 - m'_\theta s\rangle^{-d+\frac{1-\beta}{2}}$$

since $|\hat{k}\cdot x| \leq m'_\theta s$ on the support of θ_s. Combined with (A-3), this yields (A-4).

A.2 Momentum cutoff

We treat the case where $b = \theta(k/\delta)$, i.e. in momentum space.

Lemma A.4. *Let $\psi, \psi' \in \mathfrak{h}_\alpha$, then there is a $\beta > \alpha$ such that for any $\gamma, \gamma' \geq 0$ with $\gamma + \gamma' \leq \beta$;*

$$|(\psi_{s_1}, \theta(k/\delta)\psi'_{s_2})| \leq C\delta^{\gamma'}\langle s_2 - s_1\rangle^{-(2+\gamma)}. \tag{A-12}$$

Proof. Set

$$g(k) := |k|^{d-1}\theta(|k|/\delta)\overline{\hat{\psi}(k)}\hat{\psi}'(k) \tag{A-13}$$

We bound, for $n = 0, 1, 2, 3$,

$$|\partial_\omega^n g(\omega\hat{k})| \leq C\omega^{1+\beta-n}\nu(\omega/\delta), \qquad \text{with } \nu(\omega) = \sum_{j=0}^{2}|\partial^j\theta(\omega)| \tag{A-14}$$

where we wrote $\theta(|k|) = \theta(k)$ because of spherical symmetry, and we used that $\nu(\omega) = 0$ for $\omega > 1$. Since $\partial_\omega g, \partial_\omega^2 g$ are integrable in ω by the bounds (A-14), the left hand side of (A-12) is bounded by

$$C\langle s_2 - s_1\rangle^{-2}\int_{S^{d-1}} \mathrm{d}\hat{k}|f_{\hat{k}}(s_2 - s_1)|, \tag{A-15}$$

with $f_{\hat{k}}(\cdot)$ the inverse Fourier transform of $\hat{f}_{\hat{k}}(\omega) := 1_{\omega>0}\partial_\omega^2 g(\omega\hat{k})$. Furthermore, by (A-14)

$$\|\omega^{1-\gamma}\hat{f}_{\hat{k}}\|_{L^1(\mathrm{d}\omega)} + \|\omega^{-\gamma}\partial_\omega\hat{f}_{\hat{k}}\|_{L^1(\mathrm{d}\omega)} \leq C\delta^{\gamma'}, \tag{A-16}$$

uniformly in $\hat{k}$. We can now apply Lemma A.1 to the function $f_{\hat{k}}$ and we get the required bound. □

As described following Lemma A.2, the above Lemma A.4 yields items 4) and 2) of Proposition 2.5.

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